Undefined terms: point, line and plane

Every definition is made of other words. Where does geometry's chain of definitions stop, and what are the first words built from it?

WORDS TO KNOW

Undefined term: a word used as a starting point, with no definition. Geometry has three: point, line and plane.

Description: what to picture for an undefined term. It helps your intuition but is never a reason in a proof.

Coplanar: points are coplanar when one plane holds all of them.

Postulate: a rule assumed about the undefined terms. Proofs use the postulates.

Collinear: points are collinear when one line passes through all of them.

$\overleftrightarrow{AB}$**:** the line through points A and B.

KEY IDEAS

1. Three words are left undefined on purpose

A definition explains a word with other words. With only so many words, the chain must loop back or stop. Geometry stops it on purpose at three words and builds everything else on them.

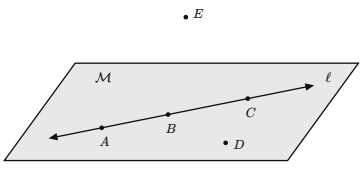
Term	Picture it as	Name it	Draw it
Point	a location with no size	a capital letter, point A	a dot
Line	perfectly straight, no thickness, no end either way	$\overleftrightarrow{AB}$ (any two of its points), or ℓ	arrowheads at both ends
Plane	perfectly flat, no thickness, no edge	$\mathcal{P}$, or three of its points not on one line, plane ABC	a slanted parallelogram

The first postulates:

- Through any two different points there is exactly one line.
- Every line contains at least two points.
- Through any three noncollinear points there is exactly one plane.
- If two different points of a line lie in a plane, the whole line lies in that plane.

2. Collinear and coplanar

The words ask for **one** line, or **one** plane, holding every point at once. In the figure, A, B and C are collinear. A, B and D are not collinear, but they are coplanar: plane $\mathcal{M}$ holds them. E is above the plane, so A, B, D and E are not coplanar.



Any two points are collinear. Any three points are coplanar.

Three noncollinear points lie in exactly one plane. Collinear points are always coplanar.

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3. Two tests for points that line up

On a coordinate grid you can check by arithmetic. A sketch cannot prove points line up.

Slope test. A, B, C are collinear exactly when the slope from A to B equals the slope from A to C. (If two of the points share an x-coordinate, all three must share it.)

Distance test. Find the longest of the three distances, say AC. The points are collinear exactly when

$$AB + BC = AC$$

and then B is between A and C. Otherwise $AB + BC > AC$.

WATCH OUT

- **"Undefined" does not mean vague.** The postulates say exactly how points, lines and planes behave. "A point has no size" is a picture to keep in mind, not a reason in a proof.
- **A line has no ends.** "The shortest path between two points" stops at those points, so it describes a segment, a piece of a line.
- **Measure both slopes from the same point.** Equal slopes with no shared point can mean two parallel lines. Simplify fractions before you compare: $\frac{6}{9}$ and $\frac{2}{3}$ are equal.
- **Compare with the longest distance.** Any other pairing can never balance. And measured lengths from a ruler are not exact, so they cannot prove points collinear.

WORKED EXAMPLES

Example 1. Line t has points D, E and F on it. Write every name the line has.

1. A line is named by any two of its points. The pairs are D and E, D and F, E and F.
2. Either order works, since a line has no direction: $\overleftrightarrow{DE}$ and $\overleftrightarrow{ED}$ are the same line. So there are 3 × 2 = 6 two-letter names.
3. Add the one-letter name t.

Answer: 7 names: $\overleftrightarrow{DE}$, $\overleftrightarrow{ED}$, $\overleftrightarrow{DF}$, $\overleftrightarrow{FD}$, $\overleftrightarrow{EF}$, $\overleftrightarrow{FE}$ and t.

Example 2. Are $P(-2, -1)$, $Q(1, 1)$ and $R(7, 5)$ collinear?

1. Slope from P to Q: $\frac{1-(-1)}{1-(-2)} = \frac{2}{3}$.
2. Slope from P to R: $\frac{5-(-1)}{7-(-2)} = \frac{6}{9} = \frac{2}{3}$.
3. The slopes are equal and both start at P, so one line through P holds Q and R.

Answer: collinear. **Check:** slope from Q to R is $\frac{5-1}{7-1} = \frac{4}{6} = \frac{2}{3}$, the same.

Example 3. (a) $AB = 5$, $BC = 8$, $AC = 13$. (b) $PQ = 7$, $QR = 11$, $PR = 15$. Are the points collinear?

1. (a) The longest is $AC = 13$. The other two add to 5 + 8 = 13. They match, so the points are collinear, and B is between A and C.
2. (b) The longest is $PR = 15$. The other two add to 7 + 11 = 18, which is more than 15. The path bends at Q.

Answer: (a) collinear, with B between A and C. (b) not collinear; P, Q, R are the corners of a triangle.

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