Axioms, definitions and theorems

Every proof answers "why?" with an earlier fact. Where do the first facts come from, and how do you know exactly what each word means?

WORDS TO KNOW

Axiom (also **postulate**): a statement a system accepts without proof and uses as a starting point.

Definition: a sentence that gives the exact test something must pass to be called by a word.

Lemma: a proved statement whose job is to help prove something else. A stepping stone.

Undefined term: a word left without a definition on purpose, such as "point" and "line". The axioms are all you know about it.

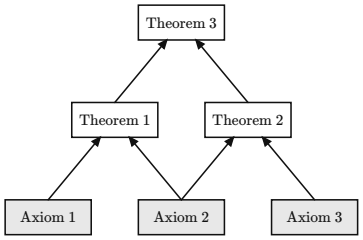
Theorem: a statement proved from the axioms and definitions. It is one of the main results.

Corollary: a proved statement that follows quickly from a result just proved.

KEY IDEAS

1. Axioms are where the reasons stop

A proof gives a reason for each step, and each reason is an earlier statement. Asking "why?" again and again cannot go on forever or go in a circle, so it has to stop somewhere. The statements where it stops are the axioms. Think of them as the rules of a game: they are chosen, not proved.

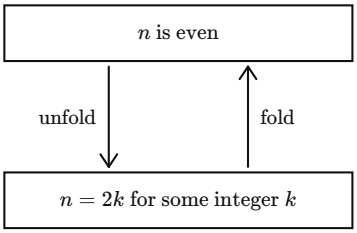


An **axiom** is accepted without proof. A **theorem** is proved from the axioms by valid steps.

To show a statement **cannot** be proved: find a world where every axiom is true and the statement is false.

2. A definition is a test that works both ways

A definition names the word, the kind of thing it is for, and the condition. For example: an integer n is **even** if $n = 2k$ for some integer k. So 0 is even: zero is 2 times the integer 0. A definition is not true or false. It is a decision about how a word is used.



- **Unfold:** if something is a W, then it meets the condition.
- **Fold:** if something meets the condition, then it is a W.

A proof with a defined word: unfold what you are given, reason, then fold to reach what you must show.

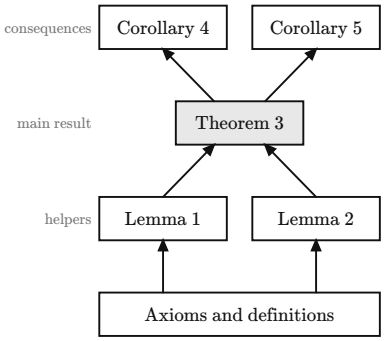
3. Theorem, lemma and corollary name jobs

All three are proved, so none is less certain than another. The word tells you what the statement is *for*. Lemmas are proved first to help, the theorem is the main result, and corollaries are the quick results that fall out of it. Any of them may be cited as a reason once it is proved.

Nothing may be cited before it is proved, and no circles: if a lemma helps prove a theorem, the lemma's proof cannot use that theorem.

A gap in a proof leaves every statement that stands on it unproved too.

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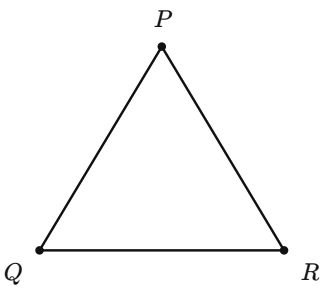


WATCH OUT

- **An axiom is not "obviously true about the real world".** It is chosen as a starting point. Change an axiom and you get a different system, not a mistake.
- **A definition gives only its own condition.** A rectangle is a quadrilateral with four right angles. Its diagonals are equal, but that is not in the definition, so it must be proved before you use it. Everyday feelings ("a rectangle is longer than it is wide") are not part of it either.
- **Each unfolding gets its own letter.** If m and n are both even, write $m = 2a$ and $n = 2b$. Using $2k$ for both would quietly say $m = n$.
- **To show a number fails a test, rule out every choice.** One value of k that does not work proves nothing. Show the equation leaves no integer choice at all.

WORKED EXAMPLES

Example 1. A world has three axioms: (1) there are exactly three points; (2) any two different points lie on exactly one line; (3) every line has exactly two points. Show it has exactly three lines.



1. By Axiom 1, name the points P, Q and R.
2. By Axiom 2, P and Q lie on one line. By Axiom 3 it holds only P and Q. Call it $\{P, Q\}$. In the same way there are lines $\{P, R\}$ and $\{Q, R\}$.
3. No fourth line: any line has two points (Axiom 3), so it holds one of the three pairs, and each pair has only one line (Axiom 2).

Answer: exactly three lines, $\{P, Q\}$, $\{P, R\}$ and $\{Q, R\}$.

Example 2. Prove that if m and n are even integers, then $m + n$ is even.

1. Unfold: $m = 2a$ and $n = 2b$ for some integers a and b (different letters).
2. Add: $m + n = 2a + 2b = 2(a + b)$.
3. Fold: $a + b$ is an integer, so $m + n$ is 2 times an integer.

Answer: $m + n$ is even, by the definition of even. **Check:** 8 + 14 = 22 = 2 · 11.

Example 3. Use the angle sum theorem (the angles of a triangle add to 180°) to show the two acute angles of a right triangle add to 90°. Then find them if one is twice the other.

1. Call the acute angles α and β. The theorem gives $\alpha + \beta + 90^\circ = 180^\circ$, so $\alpha + \beta = 90^\circ$. This is a corollary: one citation and one line.
2. Let the smaller angle be x°, so the other is $(2x)^\circ$. By the corollary, $x + 2x = 90$, so $3x = 90$ and $x = 30$.

Answer: 30° and 60°. **Check:** 30 + 60 + 90 = 180.

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