Geometry Unit 1 quiz 1: Axioms, definitions and theorems, Points, lines and planes, Lines and planes in space (answer key)

8 questions, 19 points

ANSWERS AT A GLANCE

1. Yes: $AB = DA = 5$ (they meet at A) and $BC = CD = \sqrt{13}$ (they meet at C).
2. Step (1) is correct. Step (2) is not: the equation is not part of the definition's condition, so it must be cited from a proved theorem (the Pythagorean theorem), not from the definition.
3. (a) description, (b) postulate, (c) description, (d) postulate. Only (b) and (d) may be reasons in a proof.
4. No. It fails both ways: three collinear points are coplanar, but the proposal calls them not coplanar; three points on a tabletop and one point above it are not collinear, but they are not coplanar either.
5. (3, 5), (1, 1) and (5, 3). No: the three lines meet in pairs at three different points.
6. (a) No: m has no point in P, and every point of ℓ is in P. (b) For example, with P the floor and ℓ the line where the floor meets the front wall: m where the ceiling meets the front wall is parallel to ℓ; m where the ceiling meets a side wall is skew to ℓ.
7. No. Both slopes equal $-\frac{3}{2}$, and both are measured from the shared point (1, 2), so the three points are collinear.
8. (a) The sum is $3n$ with n an integer, so it is a multiple of 3. (b) No, 200 is not a multiple of 3.

WORKED SOLUTIONS

1. A quadrilateral is a **kite** if its four sides can be split into two pairs, each pair made of two sides that meet at a corner and have the same length. Use the distance formula to decide whether the quadrilateral with corners $A(0, 4)$, $B(3, 0)$, $C(0, -2)$ and $D(-3, 0)$, joined in that order, is a kite. (2 points)

ANSWER

Yes: $AB = DA = 5$ (they meet at A) and $BC = CD = \sqrt{13}$ (they meet at C).

SOLUTION

Find the four sides.

$$AB = \sqrt{3^2 + 4^2} = 5, \qquad BC = \sqrt{3^2 + 2^2} = \sqrt{13}$$

$$CD = \sqrt{3^2 + 2^2} = \sqrt{13}, \qquad DA = \sqrt{3^2 + 4^2} = 5$$

The sides $\overline{AB}$ and $\overline{DA}$ meet at A and both have length 5. The sides $\overline{BC}$ and $\overline{CD}$ meet at C and both have length $\sqrt{13}$. These two pairs use all four sides, so $ABCD$ meets the condition and, by the definition, it is a kite.

SCORING

1 point: all four side lengths correct. 1 point: pairs $\overline{AB}$, $\overline{DA}$ and $\overline{BC}$, $\overline{CD}$ named as meeting at a corner, with the conclusion that $ABCD$ is a kite.

2. A **right triangle** is a triangle with a right angle. In $\triangle PQR$, $\angle Q$ is a right angle. A student writes two steps.

(1) $\triangle PQR$ is a right triangle. Reason: definition of right triangle.

(2) $PQ^2 + QR^2 = PR^2$. Reason: definition of right triangle.

Which step's reason is correct? For the other step, explain what is wrong and what kind of statement must be cited instead. (2 points)

ANSWER

Step (1) is correct. Step (2) is not: the equation is not part of the definition's condition, so it must be cited from a proved theorem (the Pythagorean theorem), not from the definition.

SOLUTION

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Step (1) folds the definition. The triangle has a right angle at Q, which is exactly the condition, so it may be called a right triangle, and the definition is the whole reason.

Step (2) asks the definition for more than it says. The definition's only condition is "has a right angle". The relation $PQ^2 + QR^2 = PR^2$ is true, but it is a fact about right triangles that had to be proved, so it is a theorem, the Pythagorean theorem. The step must cite that theorem. A definition gives only its own condition when it is unfolded, and nothing that has been proved about the word since.

SCORING

1 point: step (1) named correct, as the condition of the definition is met. 1 point: step (2) named wrong because the relation is not in the definition, with a proved theorem (the Pythagorean theorem) as the reason to cite.

3. Say whether each sentence is a description or a postulate. Then say which of the four may be used as a reason in a proof. (2 points)

(a) A line has no thickness.

(b) Through any two different points there is exactly one line.

(c) A plane is a flat surface that goes on without end.

(d) If two different points of a line lie in a plane, the whole line lies in that plane.

ANSWER

(a) description, (b) postulate, (c) description, (d) postulate. Only (b) and (d) may be reasons in a proof.

SOLUTION

(a) and (c) tell you what to picture: they are descriptions. (b) and (d) are rules assumed about points, lines and planes: they are postulates.

An undefined term is used in a proof only through its postulates, so (b) and (d) may be reasons, and (a) and (c) may not.

SCORING

1 point: all four sentences labeled correctly (half a point if three are right). 1 point: names (b) and (d), and only those, as reasons a proof may use.

4. Someone proposes this definition: "Points are **coplanar** when they are not collinear." Is it a good definition of coplanar? Test it in both directions, giving an example for each. (2 points)

ANSWER

No. It fails both ways: three collinear points are coplanar, but the proposal calls them not coplanar; three points on a tabletop and one point above it are not collinear, but they are not coplanar either.

SOLUTION

A definition must fit exactly the points that are coplanar, so test both directions.

Coplanar points that the proposal leaves out: three points on one line. Collinear points are always coplanar (a plane through two of them holds their whole line), but the proposal says points that are collinear are not coplanar.

Points the proposal lets in that are not coplanar: three points on a tabletop, not on one line, and a fourth point above the table. No line holds all four, so the proposal calls them coplanar. But the only plane through the three on the table is the plane of the tabletop, and the fourth point is off it, so the four are noncoplanar.

The proposal fails in both directions, so it is not a good definition.

SCORING

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1 point: a correct example of collinear points, which are coplanar but excluded by the proposal. 1 point: a correct example of noncollinear points that are noncoplanar (four or more points, one off the plane of the others), with the conclusion that the definition fails.

5. Three lines are $y = 2x - 1$, $y = -x + 8$ and $x - 2y = -1$. Find every point where two of the lines meet. Do all three lines pass through one point? (3 points)

ANSWER

(3, 5), (1, 1) and (5, 3). No: the three lines meet in pairs at three different points.

SOLUTION

No two of the lines are parallel: their slopes are 2, −1 and $\frac{1}{2}$. So each pair meets in exactly one point.

First and second: $2x - 1 = -x + 8$, so $3x = 9$, $x = 3$ and $y = 5$. The point is (3, 5).

First and third: put $y = 2x - 1$ into $x - 2y = -1$: $x - 4x + 2 = -1$, so $-3x = -3$, $x = 1$ and $y = 1$. The point is (1, 1).

Second and third: $x - 2(-x + 8) = -1$, so $3x - 16 = -1$, $x = 5$ and $y = 3$. The point is (5, 3).

The three points are different, so no point lies on all three lines (check: (3, 5) gives 3 − 10 = −7 ≠ −1 in the third equation). The lines meet in pairs at three points, the corners of a triangle.

SCORING

1 point for each pair's meeting point solved correctly (3 points), with "no" stated; take off 1 point if the answer to the last question is missing or wrong.

6. Line ℓ lies in plane P, and line m is parallel to plane P. (3 points)

(a) Can ℓ and m intersect? Explain.

(b) Using the walls, floor and ceiling of a rectangular classroom, give one example where ℓ and m are parallel and one where they are skew.

ANSWER

(a) No: m has no point in P, and every point of ℓ is in P. (b) For example, with P the floor and ℓ the line where the floor meets the front wall: m where the ceiling meets the front wall is parallel to ℓ; m where the ceiling meets a side wall is skew to ℓ.

SOLUTION

(a) No. A line parallel to a plane never meets it, so no point of m is in P. Every point of ℓ is in P. So ℓ and m share no point, and they are parallel or skew.

(b) Let P be the plane of the floor and ℓ the line where the floor meets the front wall. Any line in the ceiling is parallel to P, since the ceiling and the floor are parallel planes.

- m = the line where the ceiling meets the front wall. It lies with ℓ in the plane of the front wall and never meets it: parallel.
- m = the line where the ceiling meets a side wall. It runs front to back while ℓ runs side to side, so they are not parallel, and they lie in parallel planes, so they cannot meet: skew.

Other correct examples earn full credit.

SCORING

(a) 1 point: "no", with the reason that m has no point in P while ℓ lies in P. (b) 1 point for a correct parallel example, 1 point for a correct skew example.

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7. Mateo tests whether (−5, 11), (1, 2) and (9, −10) are collinear. He finds that the slope from the first point to the second is $\frac{-9}{6}$ and the slope from the second point to the third is $\frac{-12}{8}$. The fractions are different, so he says the points are not collinear. Is he right? Explain. (2 points)

ANSWER

No. Both slopes equal $-\frac{3}{2}$, and both are measured from the shared point (1, 2), so the three points are collinear.

SOLUTION

Simplify: $\frac{-9}{6} = -\frac{3}{2}$ and $\frac{-12}{8} = -\frac{3}{2}$. The fractions look different but are the same number.

Both slopes are taken at the shared point (1, 2), so one line through (1, 2) with slope $-\frac{3}{2}$ carries both other points. The points are collinear, and Mateo is wrong.

SCORING

1 point: shows both slopes equal $-\frac{3}{2}$. 1 point: concludes the points are collinear, using that the two slopes share the point (1, 2).

8. Take this theorem as proved: **the sum of three consecutive integers is three times the middle one**, that is, $(n-1)+n+(n+1) = 3n$ for every integer n. An integer is a **multiple of 3** if it equals $3k$ for some integer k. (3 points)

(a) Prove the corollary: the sum of any three consecutive integers is a multiple of 3.

(b) Use the corollary to decide whether 200 is the sum of three consecutive integers.

ANSWER

(a) The sum is $3n$ with n an integer, so it is a multiple of 3. (b) No, 200 is not a multiple of 3.

SOLUTION

(a) Name the middle integer n, so the three are $n-1$, n and $n+1$. By the theorem their sum is $3n$. Since n is an integer, the sum is 3 times an integer, so by the definition it is a multiple of 3. ■

(b) Suppose $200 = 3k$. Then $k = \frac{200}{3} = 66\frac{2}{3}$, which is not an integer, so 200 is not a multiple of 3. By the corollary, every sum of three consecutive integers is a multiple of 3, so 200 is not such a sum. (The nearest sums are 65 + 66 + 67 = 198 and 66 + 67 + 68 = 201.)

SCORING

1 point: (a) uses the theorem to write the sum as $3n$. 1 point: (a) folds: n is an integer, so the sum meets the definition of a multiple of 3. 1 point: (b) shows 200 is not a multiple of 3 and concludes it is not such a sum, citing the corollary.

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