Geometry Unit 1 quiz 2: The midpoint formula, Conjecture, and why a pattern is not a proof, Counterexample (answer key)

8 questions, 21 points

ANSWERS AT A GLANCE

1. $M = (-10, 1)$, $B = (-16, -1)$
2. $B = (-8, 0)$, midpoint (−1, −1)
3. Many answers; for example $A(2, 0)$, $B(6, 0)$: $M = (4, 0)$, and $OM = 4$ is not less than $OA = 2$.
4. $k = -1$ with $B = (5, 5)$, or $k = 5$ with $B = (5, -1)$
5. (a) For every triangle, at most one angle is obtuse. One triangle with two obtuse angles would disprove it; one triangle that passes settles nothing. (b) There is a rectangle whose diagonal is twice its width. True: the diagonal is $\sqrt{1+3} = 2 = 2 \cdot 1$.
6. (a) No: R is only as sure as C; it is proved only on the condition that C holds. (b) R becomes a theorem. (c) No: the proof of R falls apart, but R may still be true; it is unsettled again.
7. (a) $c = 30$. (b) A reason does, not more data: a convex hexagon splits into 4 triangles from one corner, so its angle sum is $720°$, as f says.
8. (a) Not a counterexample: the lines meet at $\left(\frac{4}{3}, -\frac{1}{3}\right)$. (b) Not a counterexample: 0 is not in the domain. (c) A counterexample: the lines are parallel and never meet.

WORKED SOLUTIONS

1. M is the midpoint of $\overline{AB}$, and N is the midpoint of $\overline{AM}$. If $A = (-4, 3)$ and $N = (-7, 2)$, find M and B. (2 points)

ANSWER

$M = (-10, 1)$, $B = (-16, -1)$

SOLUTION

N is the midpoint of $\overline{AM}$, so run the formula backwards from A through N:

$$M = (2 \cdot (-7) - (-4),\ 2 \cdot 2 - 3) = (-10, 1)$$

M is the midpoint of $\overline{AB}$, so do it again from A through M:

$$B = (2 \cdot (-10) - (-4),\ 2 \cdot 1 - 3) = (-16, -1)$$

Check: from A to N the x-coordinate changes by −3 and the y-coordinate by −1, and B is four such changes from A.

SCORING

1 point for $M = (-10, 1)$; 1 point for $B = (-16, -1)$.

2. $A = (6, -2)$, and B is a point on the x-axis. The midpoint of $\overline{AB}$ lies on the line $y = x$. Find B and the midpoint. (2 points)

ANSWER

$B = (-8, 0)$, midpoint (−1, −1)

SOLUTION

A point on the x-axis has y-coordinate 0, so write $B = (t, 0)$. The midpoint is $\left(\frac{6+t}{2}, \frac{-2+0}{2}\right) = \left(\frac{6+t}{2}, -1\right)$.

On the line $y = x$ the two coordinates are equal, so $\frac{6+t}{2} = -1$, $6 + t = -2$ and $t = -8$. So $B = (-8, 0)$ and the midpoint is (−1, −1).

Check: $\left(\frac{6+(-8)}{2}, \frac{-2+0}{2}\right) = (-1, -1)$, and −1 = −1.

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SCORING

1 point for the midpoint in terms of one unknown and the equation $\frac{6+t}{2}=-1$; 1 point for $B=(-8,0)$ and the midpoint $(-1,-1)$.

3. Claim: for every segment $\overline{AB}$ in the coordinate plane, the midpoint M of $\overline{AB}$ is closer to the origin O than A is, and closer to O than B is. Find a counterexample, and show exactly that it fails. (2 points)

ANSWER

Many answers; for example $A(2,0)$, $B(6,0)$: $M=(4,0)$, and $OM=4$ is not less than $OA=2$.

SOLUTION

The claim looks safe for segments that run past the origin, such as $A(-4,0)$ to $B(6,0)$, whose midpoint $(1,0)$ is closer to O than either end. So look at a segment that stays on one side of the origin. Take $A(2,0)$ and $B(6,0)$. The midpoint is

$$M=\left(\frac{2+6}{2},\frac{0+0}{2}\right)=(4,0)$$

Then $OM=4$ and $OA=2$, and $4<2$ is false: M is farther from O than A is. The segment meets the hypothesis and breaks the conclusion, so the claim is false.

4. $M=(1,2)$ is the midpoint of $\overline{AB}$, where $A=(-3,k)$, and $AB=10$. Find every possible value of k, and the point B that goes with each. (3 points)

ANSWER

$k=-1$ with $B=(5,5)$, or $k=5$ with $B=(5,-1)$

SOLUTION

M is the midpoint, so AM is half of AB: $AM=5$. By the distance formula,

$$(1-(-3))^2+(2-k)^2=5^2,\qquad 16+(2-k)^2=25,\qquad (2-k)^2=9$$

So $2-k=3$ or $2-k=-3$, which gives $k=-1$ or $k=5$.

Running the midpoint backwards, $B=(2\cdot 1-(-3),\ 2\cdot 2-k)=(5,4-k)$. For $k=-1$, $B=(5,5)$; for $k=5$, $B=(5,-1)$.

Check: with $k=-1$, $A=(-3,-1)$ and $AB=\sqrt{8^2+6^2}=10$; with $k=5$, $A=(-3,5)$ and $AB=\sqrt{8^2+(-6)^2}=10$.

SCORING

1 point for $AM=5$ and the equation $16+(2-k)^2=25$; 1 point for both values $k=-1$ and $k=5$; 1 point for the matching points $B=(5,5)$ and $B=(5,-1)$.

5. (3 points)

(a) Write "no triangle has two obtuse angles" as a for-every statement, naming the collection, and say what a single triangle could do to it.

(b) Write "some rectangle has a diagonal exactly twice as long as its width" as a there-is statement, and settle it using a rectangle of width 1 and length $\sqrt{3}$.

ANSWER

(a) For every triangle, at most one angle is obtuse. One triangle with two obtuse angles would disprove it; one triangle that passes settles nothing. (b) There is a rectangle whose diagonal is twice its width. True: the diagonal is $\sqrt{1+3}=2=2\cdot 1$.

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SOLUTION

(a) **For every triangle, at most one of its angles is obtuse.** It is a for-every claim, so one triangle with two obtuse angles would disprove it, and a triangle with one or no obtuse angle settles nothing.

(b) **There is a rectangle whose diagonal is twice as long as its width.** One example proves a there-is claim. With width 1 and length $\sqrt{3}$, the diagonal is the hypotenuse of a right triangle with legs 1 and $\sqrt{3}$, so it is $\sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$, which is twice the width. The claim is **true**.

SCORING

1 point: (a) a correct for-every form naming triangles as the collection. 1 point: (a) one triangle with two obtuse angles disproves it, and a passing triangle settles nothing. 1 point: (b) correct there-is form, diagonal = 2 found, and the claim called true because one example proves a there-is claim.

6. A proof of a statement R about triangles is correct in every step, except that one step uses conjecture C, which is still open. (3 points)

(a) Is R a theorem? Explain.

(b) Later, C is proved. What is the status of R now?

(c) Suppose instead that C is disproved. Does that make R false? Explain.

ANSWER

(a) No: R is only as sure as C; it is proved only on the condition that C holds. (b) R becomes a theorem. (c) No: the proof of R falls apart, but R may still be true; it is unsettled again.

SOLUTION

(a) **No.** A proof is only as sure as what it stands on, and C is a guess. What has been shown is "if C is true, then R is true." Until C is settled, R is unsettled too.

(b) Once C is a theorem, every step of the proof stands on proved results, so R is a **theorem**.

(c) **No.** Disproving C breaks this proof of R, but a broken proof does not show that R is false: R might hold for another reason. R goes back to being unsettled (a conjecture) until someone proves it another way or finds a counterexample.

SCORING

1 point: (a) no, because the proof depends on an unproved conjecture. 1 point: (b) R becomes a theorem. 1 point: (c) no, the proof fails but R is not shown false; it is unsettled.

7. The angle sums of a triangle, a quadrilateral and a pentagon are 180°, 360° and 540°. A student predicts 720° for a hexagon, following the rule $f(n) = 180(n-2)$ for a polygon with n sides. (3 points)

(a) Find a number c so that the rule $g(n) = 180(n-2) + c(n-3)(n-4)(n-5)$ also gives 180, 360, 540 at $n = 3, 4, 5$ but gives 900 at $n = 6$.

(b) The three given sums do not decide between f and g. What does? Settle the angle sum of a convex hexagon, taking the angle sum theorem for triangles as proved.

ANSWER

(a) $c = 30$. (b) A reason does, not more data: a convex hexagon splits into 4 triangles from one corner, so its angle sum is 720°, as f says.

SOLUTION

(a) The product $(n-3)(n-4)(n-5)$ is 0 at $n = 3, 4, 5$, so g agrees with f there for any c. At $n = 6$ the product is $3 \cdot 2 \cdot 1 = 6$, so $g(6) = 720 + 6c$. Setting $720 + 6c = 900$ gives $c = 30$.

(b) Both rules fit every case given, so the three sums cannot choose between them. A reason that covers the hexagon does. The diagonals from one corner of a convex hexagon split it into 4 triangles, and the

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angles of those triangles together make up the hexagon's angles. Each triangle's angles add to 180°, so the hexagon's angle sum is $4 \cdot 180° = 720°$. That matches $f(6)$, and g is wrong at $n = 6$.

SCORING

1 point: (a) $c = 30$ with the product evaluated as 6 at $n = 6$. 1 point: (b) says the given cases cannot decide, and a reason (proof) is needed. 1 point: (b) the split into 4 triangles giving 720°.

8. Claim: for every real number m other than 0, the line $y = mx + 1$ intersects the line $y = 2x - 3$. For each value of m, decide whether it is a counterexample, and say why.

(a) $m = -1$ (b) $m = 0$ (c) $m = 2$ (3 points)

ANSWER

(a) Not a counterexample: the lines meet at $\left(\frac{4}{3}, -\frac{1}{3}\right)$. (b) Not a counterexample: 0 is not in the domain. (c) A counterexample: the lines are parallel and never meet.

SOLUTION

A counterexample is a nonzero m for which the two lines do not meet.

(a) $m = -1$ is allowed. Setting $-x + 1 = 2x - 3$ gives $3x = 4$, so $x = \frac{4}{3}$ and $y = -\frac{4}{3} + 1 = -\frac{1}{3}$. The lines meet at $\left(\frac{4}{3}, -\frac{1}{3}\right)$, so the claim holds here.

(b) The claim is only about m other than 0, so $m = 0$ is outside its domain and cannot break it, whatever the lines do.

(c) $m = 2$ is allowed. The lines $y = 2x + 1$ and $y = 2x - 3$ have the same slope and different y-intercepts. Setting $2x + 1 = 2x - 3$ gives $1 = -3$, which is false for every x, so they share no point. The claim is false.

SCORING

1 point for each part with a correct reason: (a) the meeting point found or shown to exist; (b) 0 is outside the domain; (c) equal slopes and different intercepts, or the false equation $1 = -3$.

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