Geometry Unit 1 quiz 3: Inductive reasoning, Deductive reasoning, Statements and negation (answer key)

7 questions, 21 points

ANSWERS AT A GLANCE

1. "The point (3, −2) lies on the line $2x + y = 4$." True.
2. (a) All his points have two positive coordinates, and he chose them from the line $y = x$ itself. (b) One answer: (−3, 3), which is 3 from each axis but has $y \neq x$. (c) Every point the same distance from both axes lies on $y = x$ or on $y = -x$.
3. From the figure: (a), (b), (e), (g). Must be given or deduced: (c), (d), (f).
4. (a) Statement; negation "the segment does not have length 13", false. (b) Not a statement. (c) Statement; negation $\frac{5}{8} \leq \frac{2}{3}$, true. (d) Not a statement.
5. (a) The segment addition postulate needs B between A and C; that is not given. Extra given: B is between A and C. (b) The angle addition postulate only gives $m\angle POM + m\angle MOQ = 80°$; halving needs the ray to bisect the angle. Extra given: $\overrightarrow{OM}$ bisects $\angle POQ$ (or $m\angle POM = m\angle MOQ$).
6. (a) For every point P on the larger arc, $\angle APB$ has the same measure (about 50°). (b) No: a protractor reading is off by a degree or so, so they are not an exact failure. (c) No: Q is not on the larger arc, so it does not meet the hypothesis and refutes nothing about it.
7. (a) $BC = CD = DA = 5$ but $AB = 11$, so $ABCD$ is not a rhombus. (b) $EF = FG = GH = 5$, $HE = \sqrt{65}$, and no two sides are parallel, so $EFGH$ meets the repaired hypothesis and is not a rhombus: the repair is false. Excluding only the kind of figure that was found does not repair a claim.

WORKED SOLUTIONS

1. Rewrite "It is not the case that the point (3, −2) does not lie on the line $2x + y = 4$" as a sentence with no "not" in it. Is it true? Give a reason for each answer. (2 points)

ANSWER

"The point (3, −2) lies on the line $2x + y = 4$." True.

SOLUTION

The sentence is $\neg(\neg P)$, where P is "the point (3, −2) lies on the line $2x + y = 4$". By double negation, $\neg(\neg P)$ has the same truth value as P in every case, so the sentence says "the point (3, −2) lies on the line $2x + y = 4$".

Substituting, 2(3) + (−2) = 6 − 2 = 4, so the point is on the line and the sentence is true.

SCORING

1 point for the rewritten sentence with the reason (double negation); 1 point for "true" with the substitution 2(3) − 2 = 4.

2. Claim: every point that is the same distance from the x-axis as from the y-axis lies on the line $y = x$. Diego checks the points (2, 2), (5, 5) and (9, 9), and the claim holds for each. (3 points)

(a) What did Diego's choice of points have that the claim did not ask for?

(b) Give a counterexample and show exactly that it is one.

(c) Repair the claim by weakening its conclusion, and prove your repair.

ANSWER

(a) All his points have two positive coordinates, and he chose them from the line $y = x$ itself. (b) One answer: (−3, 3), which is 3 from each axis but has $y \neq x$. (c) Every point the same distance from both axes lies on $y = x$ or on $y = -x$.

SOLUTION

(a) Every point he tried has two positive coordinates (it lies in the first quadrant), and he picked points already on $y = x$. Points like that cannot break the claim, so the test was never at risk.

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(b) Take $(-3, 3)$. Its distance from the x-axis is $|3| = 3$ and from the y-axis is $|-3| = 3$, so it meets the hypothesis. On the line $y = x$ the two coordinates are equal, but $3 \neq -3$, so the point is not on the line. It is a **counterexample**.

(c) Repair: **every point the same distance from the x-axis as from the y-axis lies on the line $y = x$ or on the line $y = -x$.**

Proof. The point (x, y) is $|y|$ from the x-axis and $|x|$ from the y-axis. If $|x| = |y|$, then $y = x$ or $y = -x$, so the point lies on one of the two lines. This covers every point at once.

SCORING

(a) 1 point: names the shared feature (first quadrant or chosen on $y = x$). (b) 1 point: a point with $|x| = |y|$ and $y \neq x$, with both distances and the failure shown. (c) 1 point: the conclusion "on $y = x$ or $y = -x$" with the reason $|x| = |y|$. A repair that adds a hypothesis instead (such as "in the first quadrant") earns 0, since the question asks to weaken the conclusion.

3. A proof comes with a figure. For each fact below, say whether the proof may take it from the look of the figure, or whether it must be given or deduced. (3 points)

(a) Points A, B and C lie on one line.

(b) B is between A and C.

(c) $AB = BC$.

(d) $\angle ABD$ measures 90°.

(e) D is in the interior of $\angle ABE$.

(f) Line ℓ is parallel to line m.

(g) $\overline{AC}$ and $\overline{BD}$ cross at a point.

ANSWER

From the figure: (a), (b), (e), (g). Must be given or deduced: (c), (d), (f).

SOLUTION

A figure may be read for position: which points lie on a line, in what order, which point is inside an angle, which segments cross. So (a), (b), (e) and (g) may be taken from the figure.

It may not be read for size. Equal lengths (c), an exact angle measure such as a right angle (d), and parallel lines (f) can be drawn slightly wrong by any amount too small to see, so each must be given or deduced.

SCORING

3 points: all seven right. 2 points: five or six right. 1 point: three or four right. 0 points: fewer.

4. Decide whether each sentence is a statement. For each one that is, write its negation and give the truth value of the negation. (3 points)

(a) The segment from (0, 0) to (5, 12) has length 13.

(b) Find the midpoint of $\overline{PQ}$.

(c) $\frac{5}{8} > \frac{2}{3}$

(d) Triangles are more interesting than circles.

ANSWER

(a) Statement; negation "the segment does not have length 13", false. (b) Not a statement. (c) Statement; negation $\frac{5}{8} \leq \frac{2}{3}$, true. (d) Not a statement.

SOLUTION

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(a) A statement. The length is $\sqrt{5^2 + 12^2} = \sqrt{169} = 13$, so it is true. Its negation, "the segment from (0, 0) to (5, 12) does not have length 13", is false.

(b) Not a statement: it is an instruction.

(c) A statement. With a common denominator, $\frac{5}{8} = \frac{15}{24}$ and $\frac{2}{3} = \frac{16}{24}$, so $\frac{5}{8} > \frac{2}{3}$ is false. Its negation is $\frac{5}{8} \leq \frac{2}{3}$ (not $\frac{5}{8} < \frac{2}{3}$, which leaves out equality), and it is true.

(d) Not a statement: it is an opinion, and no fact settles it.

SCORING

1 point for sorting all four sentences correctly; 1 point for correct negations of (a) and (c), with $\leq$ in (c); 1 point for the truth values false and true.

5. Each argument has a step that is not a valid deduction. Say which accepted fact is being misused, why its if-part is not met, and what extra given would make the step valid. (4 points)

(a) A, B and C are three points with $AB = 5$ and $BC = 7$. By the segment addition postulate, $AC = 5 + 7 = 12$.

(b) M is in the interior of $\angle POQ$, and $m\angle POQ = 80°$. So $m\angle POM = 40°$.

ANSWER

(a) The segment addition postulate needs B between A and C; that is not given. Extra given: B is between A and C. (b) The angle addition postulate only gives $m\angle POM + m\angle MOQ = 80°$; halving needs the ray to bisect the angle. Extra given: $\overrightarrow{OM}$ bisects $\angle POQ$ (or $m\angle POM = m\angle MOQ$).

SOLUTION

(a) The segment addition postulate says: **if** B is between A and C, **then** $AB + BC = AC$. Nothing says the three points lie on a line with B between the others. If they form a triangle, or if A is between B and C (then $AC = 7 - 5 = 2$), AC is not 12. Adding the given "B is between A and C" makes the step valid.

(b) The fact that applies is the angle addition postulate: since M is in the interior of $\angle POQ$, $m\angle POM + m\angle MOQ = 80°$. That fixes the sum, not the two parts; 30° and 50° fit just as well. Splitting 80° evenly needs the two parts equal, which is what "$\overrightarrow{OM}$ bisects $\angle POQ$" gives. With that given, $2\,m\angle POM = 80°$ and $m\angle POM = 40°$.

SCORING

Each part 2 points: 1 point for naming the fact misused and the missing part of its if-part, 1 point for a correct extra given that makes the step valid.

6. Points A and B lie on a circle and split it into a larger arc and a smaller arc. Maya marks four points P on the larger arc and measures $\angle APB$ with a protractor each time. She reads 50°, 51°, 49° and 50°. (3 points)

(a) State the conjecture her readings suggest, precisely enough that it could be shown false.

(b) Do the readings 51° and 49° count against your conjecture? Explain.

(c) She then marks a point Q on the smaller arc and reads $m\angle AQB = 130°$. She says this refutes the conjecture. Is she right? Explain.

ANSWER

(a) For every point P on the larger arc, $\angle APB$ has the same measure (about 50°). (b) No: a protractor reading is off by a degree or so, so they are not an exact failure. (c) No: Q is not on the larger arc, so it does not meet the hypothesis and refutes nothing about it.

SOLUTION

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(a) Conjecture: **for every point P on the larger arc AB, the angle $\angle APB$ has the same measure.** The readings suggest that measure is 50°, but measuring cannot tell 50° from 50.3°, so the safer statement is that the angles are all equal. It covers infinitely many points, so four readings cannot prove it.

(b) No. A protractor reading is off by a degree or so, and 49°, 50° and 51° are all within that of each other. A counterexample's failure must be exact; these readings are not evidence either way about a difference of 1°.

(c) No. The conjecture is about points on the larger arc, and Q is on the smaller arc, so Q does not meet the hypothesis. A figure that misses a hypothesis proves nothing. The reading would refute a different claim, "every point of the circle gives the same angle": 130° against 50° is far too big a gap to be measuring error. That is why stating the conjecture precisely, with "on the larger arc", mattered.

SCORING

(a) 1 point: a for-every statement that names the points (on the larger arc) and the conclusion (equal angles, or 50°). A statement that does not limit P to the larger arc earns 0. (b) 1 point: "no", with measurement error as the reason. (c) 1 point: "no", because Q does not meet the hypothesis. A "yes" earns 0 unless the student's own conjecture in (a) was about every point of the circle and they say so.

7. Claim: a quadrilateral with three sides of equal length is a rhombus. (A rhombus is a quadrilateral with four sides of equal length.) (3 points)

(a) Show exactly that $A(0,0)$, $B(11,0)$, $C(8,4)$, $D(3,4)$ is a counterexample.

(b) Owen notices that $ABCD$ has a pair of parallel sides, and repairs the claim: a quadrilateral with three sides of equal length and no pair of parallel sides is a rhombus. Test his repair on the convex quadrilateral $E(0,0)$, $F(5,0)$, $G(8,4)$, $H(4,7)$, and say what the result shows about his way of repairing.

ANSWER

(a) $BC = CD = DA = 5$ but $AB = 11$, so $ABCD$ is not a rhombus. (b) $EF = FG = GH = 5$, $HE = \sqrt{65}$, and no two sides are parallel, so $EFGH$ meets the repaired hypothesis and is not a rhombus: the repair is false. Excluding only the kind of figure that was found does not repair a claim.

SOLUTION

(a) $BC = \sqrt{3^2 + 4^2} = 5$, $CD = 8 - 3 = 5$ and $DA = \sqrt{3^2 + 4^2} = 5$, so three sides are equal and the hypothesis is met. $AB = 11 \neq 5$, so the four sides are not all equal and $ABCD$ is not a rhombus. It is a **counterexample**.

(b) $EF = 5$, $FG = \sqrt{3^2 + 4^2} = 5$, $GH = \sqrt{4^2 + 3^2} = 5$ and $HE = \sqrt{4^2 + 7^2} = \sqrt{65}$. The slopes of $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, $\overline{HE}$ are 0, $\frac{4}{3}$, $-\frac{3}{4}$ and $\frac{7}{4}$, all different, so no two sides are parallel. $EFGH$ meets Owen's hypothesis, and $HE \neq 5$, so it is not a rhombus. The repair is **false**.

Owen shut out the feature of the one counterexample he had, but the claim fails for a reason that does not need parallel sides: three equal sides do not fix the fourth. Every figure like $EFGH$ was still inside his claim.

SCORING

(a) 1 point: the three equal sides and the unequal fourth side, each computed. (b) 2 points: 1 for showing $EFGH$ meets the hypothesis (three sides of 5 and no parallel pair, by slopes); 1 for concluding the repair is false and saying that it only excluded the kind of figure found.

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