Geometry Unit 1 quiz 4: Conditionals, converse, inverse, contrapositive, Biconditional, Converses and biconditional definitions (answer key)

7 questions, 20 points

ANSWERS AT A GLANCE

1. P false, Q true; $P \to Q$ true, converse false, inverse false, contrapositive true.
2. (a) row 2, false (b) row 4, true (c) row 1, true (d) row 3, false
3. (a) For example $(4, -5)$: $x > 3$ but $x + y = -1$. (b) If $x + y \leq 3$, then $x \leq 3$ and $y \leq 3$. False, broken by the same point $(4, -5)$. (c) If $x + y > 3$, then $x > 3$ or $y > 3$. False: $(2, 2)$ has $x + y = 4$ with neither coordinate above 3.
4. (a) If B lies between A and C, then $AB + BC = AC$. True. (b) If two different planes share a point, then they share a whole line. True. (c) If two lines in space do not intersect, then they are skew. False: two parallel lines, such as two opposite edges of the top face of a box.
5. (a) Holds. (b) Fails: $\angle ABD = 30^\circ$ and $\angle DBC = 40^\circ$ with D inside $\angle ABC$ share vertex B and side $\overrightarrow{BD}$, but $m\angle ABC = 70^\circ$, so $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are not opposite rays. (c) Too loose; it leaves out "their other sides are opposite rays".
6. Converse: if each of the two angles is acute, then they are complementary (false). Inverse: if the two angles are not complementary, then at least one of them is not acute (false). Contrapositive: if at least one of the two angles is not acute, then they are not complementary (true). The original is true. Angles of $30°$ and $40°$ break the converse and the inverse.
7. (i) definition of angle bisector, forward (ii) definition of congruent angles, forward (iii) definition of angle bisector, forward (iv) definition of right angle, backward

WORKED SOLUTIONS

1. Points A, B and C lie on a line in that order, with $AB = 5$ and $BC = 7$. Let P be "B is the midpoint of $\overline{AC}$" and let Q be "$AC = 12$". Give the truth value of P, of Q, of $P \to Q$, and of its converse, inverse and contrapositive. (2 points)

ANSWER

P false, Q true; $P \to Q$ true, converse false, inverse false, contrapositive true.

SOLUTION

$AB = 5$ and $BC = 7$ are not equal, so B is not the midpoint of $\overline{AC}$: P is false. B is between A and C, so by the segment addition postulate $AC = AB + BC = 5 + 7 = 12$: Q is true.

- $P \to Q$: hypothesis P false, so **true** (vacuously).
- Converse $Q \to P$: hypothesis true, conclusion false, so **false**.
- Inverse $\neg P \to \neg Q$: $\neg P$ true and $\neg Q$ false, so **false**, matching the converse.
- Contrapositive $\neg Q \to \neg P$: hypothesis $\neg Q$ false, so **true** (vacuously), matching $P \to Q$.

2. Let S be the statement "the distance from $(a, 0)$ to $(0, 4)$ is 5 if and only if $a > 0$." For each value of a, name the row of the table S falls on and say whether S is true.

(a) $a = -3$ (b) $a = 0$ (c) $a = 3$ (d) $a = 5$ (2 points)

ANSWER

(a) row 2, false (b) row 4, true (c) row 1, true (d) row 3, false

SOLUTION

The distance is $\sqrt{a^2 + 16}$, which is 5 exactly when $a^2 = 9$.

(a) $\sqrt{9 + 16} = 5$, so the first part is true; $-3 > 0$ is false. T, F: **row 2, false**.

(b) $\sqrt{16} = 4$, so the first part is false; $0 > 0$ is false. F, F: **row 4, true**.

(c) $\sqrt{9 + 16} = 5$ and $3 > 0$. T, T: **row 1, true**.

(d) $\sqrt{25 + 16} = \sqrt{41} \neq 5$, but $5 > 0$. F, T: **row 3, false**.

SCORING

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1 point for the four distances judged correctly (all four parts' first part), 1 point for all four rows and truth values correct. Take off the second point for any row 4 marked false.

3. Claim: "For every point (x, y), if $x > 3$ or $y > 3$, then $x + y > 3$." (3 points)

(a) Show that the claim is false with a counterexample.

(b) Write the contrapositive of the claim, negating the "or" correctly. Is the contrapositive true?

(c) Write the converse, and decide whether it is true.

ANSWER

(a) For example (4, −5): $x > 3$ but $x + y = -1$. (b) If $x + y \leq 3$, then $x \leq 3$ and $y \leq 3$. False, broken by the same point (4, −5). (c) If $x + y > 3$, then $x > 3$ or $y > 3$. False: (2, 2) has $x + y = 4$ with neither coordinate above 3.

SOLUTION

(a) Take (4, −5). The hypothesis holds, since 4 > 3, but $x + y = 4 + (-5) = -1$, which is not more than 3. So the claim is false.

(b) The negation of "$x + y > 3$" is "$x + y \leq 3$", and the negation of "$x > 3$ or $y > 3$" is "$x \leq 3$ and $y \leq 3$". Contrapositive: *if* $x + y \leq 3$*, then* $x \leq 3$ *and* $y \leq 3$. It is equivalent to the claim, so it is false too; (4, −5) breaks it, since −1 ≤ 3 but 4 > 3.

(c) Converse: *if* $x + y > 3$*, then* $x > 3$ *or* $y > 3$. False: (2, 2) has $x + y = 4 > 3$, but 2 ≤ 3 and 2 ≤ 3.

SCORING

1 point: a valid counterexample to the claim, checked. 1 point: the contrapositive with "or" turned into "and" and both inequalities negated, judged false. 1 point: the converse written and shown false with a checked counterexample.

4. Rewrite each sentence as an if-then statement. Then decide whether it is true, and give a counterexample if it is false. (3 points)

(a) For points A, B and C on a line, B lying between A and C is sufficient for $AB + BC = AC$.

(b) Two different planes share a point only if they share a whole line.

(c) For two lines in space to be skew, it is sufficient that they do not intersect.

ANSWER

(a) If B lies between A and C, then $AB + BC = AC$. True. (b) If two different planes share a point, then they share a whole line. True. (c) If two lines in space do not intersect, then they are skew. False: two parallel lines, such as two opposite edges of the top face of a box.

SOLUTION

(a) The sufficient condition goes after "if": *if* B *lies between* A *and* C*, then* $AB + BC = AC$. **True**, by the segment addition postulate.

(b) "Only if" introduces the necessary condition, which goes after "then": *if two different planes share a point, then they share a whole line*. **True**: two different planes that meet, meet in a line.

(c) The sufficient condition goes after "if": *if two lines in space do not intersect, then they are skew*. **False**: two opposite edges of the top face of a box lie on parallel lines. The lines never meet, but they lie in one plane, so they are not skew.

SCORING

1 point per part: the if-then with hypothesis and conclusion in the right places, and the correct truth value (with the counterexample in (c)). No point for a part whose if-then is reversed, even with a correct truth value.

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5. Two angles form a **linear pair** when they share a side and their other sides are opposite rays. Someone proposes instead: "Two angles form a linear pair if they share a vertex and a side." (3 points)

(a) Test the forward direction of the proposal.

(b) Test the backward direction. If it fails, give a counterexample with angle measures.

(c) Is the proposal too loose, too tight or both? Which part of the true definition does it leave out?

ANSWER

(a) Holds. (b) Fails: $\angle ABD = 30^\circ$ and $\angle DBC = 40^\circ$ with D inside $\angle ABC$ share vertex B and side $\overrightarrow{BD}$, but $m\angle ABC = 70^\circ$, so $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are not opposite rays. (c) Too loose; it leaves out "their other sides are opposite rays".

SOLUTION

(a) Forward: if two angles form a linear pair, they share a side, and a shared side starts at the shared vertex. So they share a vertex and a side. This direction holds.

(b) Backward: let D be in the interior of $\angle ABC$, with $m\angle ABD = 30^\circ$ and $m\angle DBC = 40^\circ$. The two angles share the vertex B and the side $\overrightarrow{BD}$, so they meet the proposed condition. By the angle addition postulate, $m\angle ABC = 30^\circ + 40^\circ = 70^\circ$. Opposite rays would form a straight angle of 180°, so $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are not opposite rays, and the angles do not form a linear pair. This direction fails.

(c) Something that meets the condition is not a linear pair, so the proposal is **too loose**. It drops the clause "their other sides are opposite rays".

SCORING

1 point for (a) with a reason. 1 point for (b): a pair of angles that share a vertex and a side, with measures showing the other sides are not opposite rays. 1 point for (c): "too loose" and the missing clause named.

6. For two angles, each with a positive measure, take the statement: "If the two angles are complementary, then each of them is acute." Write its converse, inverse and contrapositive. Say which of the four statements are true, and give a counterexample to each false one. (3 points)

ANSWER

Converse: if each of the two angles is acute, then they are complementary (false). Inverse: if the two angles are not complementary, then at least one of them is not acute (false). Contrapositive: if at least one of the two angles is not acute, then they are not complementary (true). The original is true. Angles of 30° and 40° break the converse and the inverse.

SOLUTION

Original: true. If $m\angle 1 + m\angle 2 = 90^\circ$ and both measures are positive, each measure is less than 90°, so each angle is acute.

The negation of "each of them is acute" is "at least one of them is not acute".

- Converse: *if each of the two angles is acute, then they are complementary*. **False**: 30° and 40° are both acute, but 30° + 40° = 70° ≠ 90°.
- Inverse: *if the two angles are not complementary, then at least one of them is not acute*. **False**: the same 30° and 40° are not complementary, yet both are acute.
- Contrapositive: *if at least one of the two angles is not acute, then they are not complementary*. **True**: it is equivalent to the original. (Directly: an angle that is not acute measures 90° or more, and the other is positive, so the sum is more than 90°.)

SCORING

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1 point: converse, inverse and contrapositive all formed correctly, with "each is acute" negated as "at least one is not acute". 1 point: original and contrapositive true, converse and inverse false. 1 point: a correct counterexample (two acute angles whose measures do not add to 90°) shown to break the converse and the inverse.

7. **Given:** $\overrightarrow{OM}$ bisects $\angle LON$, and $m\angle LOM = 45°$. **Prove:** $\angle LON$ is a right angle.

Jin's proof has four reasons missing.

	Statement	Reason
1	$\overrightarrow{OM}$ bisects $\angle LON$	given
2	$\angle LOM \cong \angle MON$	(i)
3	$m\angle MON = m\angle LOM$	(ii)
4	$m\angle LOM = 45°$	given
5	$m\angle MON = 45°$	substitution
6	M is in the interior of $\angle LON$	(iii)
7	$m\angle LON = m\angle LOM + m\angle MON$	angle addition postulate
8	$m\angle LON = 90°$	substitution
9	$\angle LON$ is a right angle	(iv)

Give reasons (i) to (iv). Each is a definition: say whether it runs forward or backward. (4 points)

ANSWER

(i) definition of angle bisector, forward (ii) definition of congruent angles, forward (iii) definition of angle bisector, forward (iv) definition of right angle, backward

SOLUTION

(i) Line 1 hands us the word "bisects", and line 2 is part of what it means: definition of angle bisector, **forward**.

(ii) Line 2 hands us "congruent", and line 3 is what congruent angles have, equal measures: definition of congruent angles, **forward**.

(iii) The other part of what "bisects" means is that M is in the interior. This is what lets line 7 use the angle addition postulate. Definition of angle bisector, **forward**.

(iv) Line 8 checks the condition, a measure of 90°, and line 9 earns the word: definition of right angle, **backward**.

The definition of angle bisector is used twice, both times forward, since the proof starts from that word.

SCORING

1 point for each of (i) to (iv): the correct definition and the correct direction. A right definition with the wrong direction earns nothing for that line.

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