Geometry Unit 1 test: Points, lines, planes and the language of proof (answer key)

12 questions, 49 points

ANSWERS AT A GLANCE

1. (a) If two lines intersect, then they are perpendicular. False: two lines that cross at a 60° angle form angles of 60° and 120°, none of them right. (b) If two lines form four right angles, then they are perpendicular. True, by the definition of perpendicular lines (run backward).
2. (a) $AB = 5$, $BC = 10$, $AC = 15$. (b) Yes: $AB + BC = AC$, so B lies on $\overline{AC}$. (c) No: the slope from A to D is $\frac{7}{8}$ but from A to C it is $\frac{3}{4}$. So $AD + DC > AC$.
3. (a) $M = (-1, 5)$, $N = (1, 1)$. (b) Both slopes are −2; $MN = 2\sqrt{5}$ and $QR = 4\sqrt{5}$.
4. (a) The midpoint is $(2, 2)$ and its distance to A is 3. (b) For example $P = (-1, 5)$. (c) The only point with $PA = 3$ and $PB = 3$ is $(2, 2)$, the midpoint.
5. (a) For example $P(0, 0)$, $Q(3, 4)$: $PQ = 5$, but $|3 - 0| + |4 - 0| = 7$. (b) Exactly when $x_1 = x_2$ or $y_1 = y_2$: the segment is vertical or horizontal (or $P = Q$).
6. (a) $m_1 m_2 = -1$ forces $m_1 \neq m_2$, and two lines with different slopes intersect. (b) Converse: if the two lines intersect, then $m_1 m_2 = -1$. Inverse: if $m_1 m_2 \neq -1$, then the two lines do not intersect. Both are broken by $y = x$ and $y = 2x$, which meet at $(0, 0)$ with $m_1 m_2 = 2$. (c) The contrapositive gives only $-\frac{1}{4}a \neq -1$, that is, $a \neq 4$. In fact $a = -\frac{1}{4}$.
7. (a) $y = \frac{4}{3}x - \frac{5}{3}$ and $y = -\frac{3}{4}x + \frac{5}{2}$; they share only $(2, 1)$. (b) Both midpoints are $(2, 1)$. (c) Yes: it passes through the midpoint $(2, 1)$ of $\overline{AB}$, and it is perpendicular to $\overline{AB}$ (the slopes multiply to −1).
8. (a) Longest side $\overline{BC} = 10$, midpoint $(3, 4)$, which is 5 from each corner: the case passes. (b) Longest side $\overline{DE} = 8$, midpoint $(4, 0)$, which is 4 from D and E but $\sqrt{20}$ from F: the conjecture is false. (c) "In every right triangle, the midpoint of the longest side is the same distance from all three corners." It is a new conjecture until it is proved.
9. (a) $AB = 6$, $AC = 5$, $BC = 5$; P false, Q true. (b) "Triangle ABC is not equilateral." (c) Column F, F, F, T; row 3; false.
10. (a) Yes: $AB = BC = 5$ and $AC = \sqrt{50}$. (b) $FD = FE = \sqrt{9 + t^2}$. (c) $t = 3\sqrt{3}$; isosceles under "at least two", not isosceles under "exactly two".
11. (a) For $P(2, 1)$ both sums are 66; for $P(5, 4)$ both are 54. (b) For every rectangle $ABCD$ and every point P inside it, $PA^2 + PC^2 = PB^2 + PD^2$. (c) Both sums equal $x^2 + y^2 + (x - w)^2 + (y - h)^2$. (d) No: with $P(3, 1)$ the sums are 20 and 4. The proof used the right angles of the rectangle, in placing its corners.
12. See the proof below.

WORKED SOLUTIONS

1. Two lines are **perpendicular** if they meet to form a right angle. Both statements below are about two different lines in a plane, and both are true. (2 points)

   (a) If two lines are perpendicular, then they intersect.

   (b) If two lines are perpendicular, then all four angles they form are right angles.

   Write the converse of each statement. Say whether the converse is true, and back up your answer: name the reason that proves it, or give a counterexample.

   ANSWER

   (a) If two lines intersect, then they are perpendicular. False: two lines that cross at a 60° angle form angles of 60° and 120°, none of them right.

   (b) If two lines form four right angles, then they are perpendicular. True, by the definition of perpendicular lines (run backward).

   SOLUTION

   The converse swaps the if-part and the then-part, keeping the setting.

   (a) Converse: if two lines intersect, then they are perpendicular. A counterexample must meet the new if-part and fail the new then-part. Take two lines that cross so that one of the angles measures 60°. Its neighbor forms a linear pair with it, so it measures 180° − 60° = 120°, and the other two angles are

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vertical to these, so the four angles are 60°, 120°, 60° and 120°. The lines intersect, but they do not meet to form a right angle, so they are not perpendicular. The converse is **false**.

(b) Converse: if two lines form four right angles, then they are perpendicular. If all four angles are right angles, then in particular the lines meet to form a right angle, and that is the condition in the definition. By the definition of perpendicular lines, run backward, the lines are perpendicular. The converse is **true**.

Both statements were true, yet one converse is true and the other false. A true statement says nothing about its converse; each direction needs its own proof or its own counterexample.

SCORING

1 point for (a): the converse written correctly, called false, and a valid counterexample (two lines that cross with no right angle). 1 point for (b): the converse written correctly, called true, and the definition of perpendicular lines named. A verdict with no counterexample or reason earns nothing for that part.

2. The points are $A(-3, -2)$, $B(1, 1)$ and $C(9, 7)$. (3 points)

(a) Use the distance formula to find AB, BC and AC.

(b) Use the distance test to decide whether B lies on $\overline{AC}$.

(c) The point $D(5, 5)$ is also given. Without the distance formula, decide whether D lies on the line through A and C. Then say whether $AD + DC$ equals AC or is greater than it, and why.

ANSWER

(a) $AB = 5$, $BC = 10$, $AC = 15$. (b) Yes: $AB + BC = AC$, so B lies on $\overline{AC}$. (c) No: the slope from A to D is $\frac{7}{8}$ but from A to C it is $\frac{3}{4}$. So $AD + DC > AC$.

SOLUTION

(a) $AB = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$, $BC = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$ and $AC = \sqrt{12^2 + 9^2} = \sqrt{225} = 15$.

(b) The longest distance is AC, and $AB + BC = 5 + 10 = 15 = AC$. So the points are collinear and B lies between A and C, on $\overline{AC}$.

(c) Use the slope test from the shared point A. From A to C the slope is $\frac{7 - (-2)}{9 - (-3)} = \frac{9}{12} = \frac{3}{4}$. From A to D it is $\frac{5 - (-2)}{5 - (-3)} = \frac{7}{8}$. The slopes differ, so D is not on the line through A and C.

By the distance test, $AD + DC = AC$ holds only when D lies on $\overline{AC}$. D is not even on the line, so A, D, C are the corners of a triangle and $AD + DC > AC$. (Check: $AD = \sqrt{113} \approx 10.63$ and $DC = \sqrt{20} \approx 4.47$, which add to about 15.10.)

SCORING

1 point: (a) all three distances. 1 point: (b) the comparison $AB + BC = AC$ with the longest distance, concluding B is on $\overline{AC}$. 1 point: (c) the slopes $\frac{7}{8}$ and $\frac{3}{4}$ from A, so D is off the line, and $AD + DC > AC$ because equality needs D on $\overline{AC}$.

3. Triangle PQR has $P = (-4, 3)$, $Q = (2, 7)$ and $R = (6, -1)$. M is the midpoint of $\overline{PQ}$ and N is the midpoint of $\overline{PR}$. (3 points)

(a) Find M and N.

(b) Show that $\overline{MN}$ is parallel to $\overline{QR}$ and half as long.

ANSWER

(a) $M = (-1, 5)$, $N = (1, 1)$. (b) Both slopes are −2; $MN = 2\sqrt{5}$ and $QR = 4\sqrt{5}$.

SOLUTION

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(a) $M = \left(\frac{-4+2}{2}, \frac{3+7}{2}\right) = (-1, 5)$ and $N = \left(\frac{-4+6}{2}, \frac{3+(-1)}{2}\right) = (1, 1)$.

(b) The slope of $\overline{MN}$ is $\frac{1-5}{1-(-1)} = \frac{-4}{2} = -2$, and the slope of $\overline{QR}$ is $\frac{-1-7}{6-2} = \frac{-8}{4} = -2$. Equal slopes, so the segments are parallel.

By the distance formula, $MN = \sqrt{2^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}$ and $QR = \sqrt{4^2 + (-8)^2} = \sqrt{80} = 4\sqrt{5}$. So $MN = \frac{1}{2}QR$.

SCORING

1 point for $M = (-1, 5)$ and $N = (1, 1)$; 1 point for both slopes equal to −2; 1 point for $MN = 2\sqrt{5}$ and $QR = 4\sqrt{5}$, so MN is half of QR.

4. Let $A = (-1, 2)$ and $B = (5, 2)$. For a point P, consider the statement "P is the midpoint of $\overline{AB}$ if and only if $PA = 3$." (3 points)

(a) Show that the forward direction holds.

(b) Show that the backward direction fails, with a point P that has whole-number coordinates.

(c) Replace "$PA = 3$" by "$PA = 3$ and $PB = 3$". Show that the new statement is true for every point P.

ANSWER

(a) The midpoint is (2, 2) and its distance to A is 3. (b) For example $P = (-1, 5)$. (c) The only point with $PA = 3$ and $PB = 3$ is (2, 2), the midpoint.

SOLUTION

(a) The midpoint of $\overline{AB}$ is $\left(\frac{-1+5}{2}, \frac{2+2}{2}\right) = (2, 2)$, and its distance to A is $\sqrt{3^2 + 0^2} = 3$. So if P is the midpoint, then $PA = 3$.

(b) Take $P = (-1, 5)$. Then $PA = \sqrt{0^2 + 3^2} = 3$, but P is not (2, 2). Here $PA = 3$ is true and "P is the midpoint" is false, so the backward direction fails. ((−4, 2) also works.)

(c) Forward: the midpoint (2, 2) has $PA = 3$ by (a) and $PB = \sqrt{3^2 + 0^2} = 3$.

Backward: let $P = (x, y)$ with $PA = 3$ and $PB = 3$. Then $(x+1)^2 + (y-2)^2 = 9$ and $(x-5)^2 + (y-2)^2 = 9$. Subtract the second from the first: $(x+1)^2 - (x-5)^2 = 0$, that is $12x - 24 = 0$, so $x = 2$. Then $9 + (y-2)^2 = 9$, so $y = 2$. The only such point is (2, 2), the midpoint. Both directions hold, so the new statement is true for every point.

SCORING

1 point for (a) with the midpoint found. 1 point for (b): a correct point with $PA = 3$ checked. 1 point for (c): the backward direction solved to the single point (2, 2) (the forward direction alone earns nothing).

5. Claim: for all points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the distance $PQ = |x_2 - x_1| + |y_2 - y_1|$. (4 points)

(a) Give a counterexample, and work out both sides exactly.

(b) Find every pair of points P, Q for which the equation does hold, and show why.

ANSWER

(a) For example $P(0, 0)$, $Q(3, 4)$: $PQ = 5$, but $|3 - 0| + |4 - 0| = 7$. (b) Exactly when $x_1 = x_2$ or $y_1 = y_2$: the segment is vertical or horizontal (or $P = Q$).

SOLUTION

(a) Take $P(0, 0)$ and $Q(3, 4)$. By the distance formula

$$PQ = \sqrt{(3-0)^2 + (4-0)^2} = \sqrt{25} = 5$$

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while $|3-0|+|4-0| = 3+4 = 7$. Since $5 \neq 7$, the claim is false.

(b) Write $a = |x_2 - x_1|$ and $b = |y_2 - y_1|$, both ≥ 0. By the distance formula $PQ = \sqrt{a^2+b^2}$, so the question is when $\sqrt{a^2+b^2} = a+b$. Both sides are ≥ 0, so they are equal exactly when their squares are:

$$a^2+b^2 = a^2+2ab+b^2 \iff 2ab = 0 \iff a = 0 \text{ or } b = 0$$

So the equation holds exactly when $x_1 = x_2$ (a vertical segment) or $y_1 = y_2$ (a horizontal segment), including $P = Q$. For every other pair of points it fails, so the counterexample in (a) was not a rare case.

SCORING

(a) 2 points: 1 for a pair of points not on one horizontal or vertical line, 1 for both sides computed exactly and compared. (b) 2 points: 1 for the answer "vertical or horizontal segments", 1 for a reason that covers every pair (squaring both sides to get $2ab = 0$, or an equivalent argument). A list of examples alone earns no reason point.

6. Two non-vertical lines in the coordinate plane have slopes m_1 and m_2. Statement S: "If $m_1 m_2 = -1$, then the two lines intersect." (4 points)

(a) Explain why S is true.

(b) Write the converse and the inverse of S, and show that both are false with one pair of lines.

(c) The lines $y = ax + 2$ and $y = -\frac{1}{4}x + 5$ do not intersect. Write what the contrapositive of S tells you about a. Then find a exactly.

ANSWER

(a) $m_1 m_2 = -1$ forces $m_1 \neq m_2$, and two lines with different slopes intersect. (b) Converse: if the two lines intersect, then $m_1 m_2 = -1$. Inverse: if $m_1 m_2 \neq -1$, then the two lines do not intersect. Both are broken by $y = x$ and $y = 2x$, which meet at (0, 0) with $m_1 m_2 = 2$. (c) The contrapositive gives only $-\frac{1}{4}a \neq -1$, that is, $a \neq 4$. In fact $a = -\frac{1}{4}$.

SOLUTION

(a) If $m_1 = m_2$, then $m_1 m_2 = m_1^2$, which is never negative, so it cannot equal –1. So $m_1 m_2 = -1$ forces $m_1 \neq m_2$. Two lines in a plane with different slopes are not parallel, so they meet (in exactly one point). Hence S is true.

(b) Converse: *if the two lines intersect, then* $m_1 m_2 = -1$. Inverse: *if* $m_1 m_2 \neq -1$*, then the two lines do not intersect.* Take $y = x$ and $y = 2x$. They meet at (0, 0), and $m_1 m_2 = 1 \cdot 2 = 2 \neq -1$. For the converse the hypothesis (they intersect) is true and the conclusion false; for the inverse the hypothesis ($m_1 m_2 \neq -1$) is true and the conclusion (they do not intersect) false. One pair breaks both because the converse and the inverse are logically equivalent.

(c) Contrapositive of S: *if the two lines do not intersect, then* $m_1 m_2 \neq -1$. Here the lines do not intersect, so $a \cdot \left(-\frac{1}{4}\right) \neq -1$, that is, $a \neq 4$. That is all the contrapositive gives. To find a, use what non-intersecting lines in a plane are: parallel, with equal slopes. So $a = -\frac{1}{4}$ (and the lines are different, since their y-intercepts 2 and 5 differ).

SCORING

1 point: (a) shows $m_1 m_2 = -1$ rules out equal slopes, so the lines meet. 1 point: converse and inverse both written correctly. 1 point: one pair of intersecting lines with $m_1 m_2 \neq -1$, shown to break both. 1 point: (c) the contrapositive gives $a \neq 4$, and equal slopes give $a = -\frac{1}{4}$.

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7. The points are $A(-1,-3)$, $B(5,5)$, $C(-2,4)$ and $D(6,-2)$. (5 points)

(a) Write equations of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, and find every point the two lines share. Explain why there is no other.

(b) Show that this point is the midpoint of $\overline{AB}$ and of $\overline{CD}$.

(c) Is $\overleftrightarrow{CD}$ the perpendicular bisector of $\overline{AB}$? Explain.

ANSWER

(a) $y = \frac{4}{3}x - \frac{5}{3}$ and $y = -\frac{3}{4}x + \frac{5}{2}$; they share only (2, 1). (b) Both midpoints are (2, 1). (c) Yes: it passes through the midpoint (2, 1) of $\overline{AB}$, and it is perpendicular to $\overline{AB}$ (the slopes multiply to −1).

SOLUTION

(a) Slope of $\overleftrightarrow{AB}$: $\frac{5-(-3)}{5-(-1)} = \frac{8}{6} = \frac{4}{3}$, so $y + 3 = \frac{4}{3}(x+1)$, which is $y = \frac{4}{3}x - \frac{5}{3}$.

Slope of $\overleftrightarrow{CD}$: $\frac{-2-4}{6-(-2)} = \frac{-6}{8} = -\frac{3}{4}$, so $y - 4 = -\frac{3}{4}(x+2)$, which is $y = -\frac{3}{4}x + \frac{5}{2}$.

Set them equal and multiply by 12: $16x - 20 = -9x + 30$, so $25x = 50$, $x = 2$ and $y = \frac{8}{3} - \frac{5}{3} = 1$. Check: $-\frac{3}{2} + \frac{5}{2} = 1$. The slopes differ, so the lines are different, and two different lines meet in at most one point: (2, 1) is the only shared point.

(b) Midpoint of $\overline{AB}$: $\left(\frac{-1+5}{2}, \frac{-3+5}{2}\right) = (2,1)$. Midpoint of $\overline{CD}$: $\left(\frac{-2+6}{2}, \frac{4+(-2)}{2}\right) = (2,1)$.

(c) Yes. $\overleftrightarrow{CD}$ passes through (2, 1), the midpoint of $\overline{AB}$, and $\frac{4}{3} \times \left(-\frac{3}{4}\right) = -1$, so it is perpendicular to $\overline{AB}$. (The same follows from the moves: A to B is 6 right and 8 up, and C to D is 8 right and 6 down, the first move turned a quarter turn.) A line through the midpoint and perpendicular to the segment is its perpendicular bisector.

SCORING

(a) 1 point for both equations; 1 point for (2, 1) with the reason that two different lines meet at most once. (b) 1 point for both midpoints. (c) 1 point for passing through the midpoint, 1 point for showing the lines are perpendicular (slope product −1, or the quarter-turn of the move from A to B).

8. Conjecture: in every triangle, the midpoint of the longest side is the same distance from all three corners. (5 points)

(a) Test the conjecture on the triangle with $A(0,0)$, $B(6,0)$ and $C(0,8)$.

(b) Test it on the triangle with $D(0,0)$, $E(8,0)$ and $F(2,4)$, and settle the conjecture.

(c) Triangle ABC has a right angle and triangle DEF does not. Use this to repair the conjecture, and say what the status of the repaired claim is.

ANSWER

(a) Longest side $\overline{BC} = 10$, midpoint (3, 4), which is 5 from each corner: the case passes. (b) Longest side $\overline{DE} = 8$, midpoint (4, 0), which is 4 from D and E but $\sqrt{20}$ from F: the conjecture is false. (c) "In every right triangle, the midpoint of the longest side is the same distance from all three corners." It is a new conjecture until it is proved.

SOLUTION

(a) $AB = 6$, $AC = 8$ and $BC = \sqrt{6^2 + 8^2} = 10$, so $\overline{BC}$ is longest. Its midpoint is $M = \left(\frac{6+0}{2}, \frac{0+8}{2}\right) = (3,4)$. Then $MA = \sqrt{9+16} = 5$, $MB = \sqrt{(6-3)^2 + (0-4)^2} = 5$ and $MC = \sqrt{(0-3)^2 + (8-4)^2} = 5$. The case passes, which settles nothing.

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(b) $DE = 8$, $DF = \sqrt{4 + 16} = \sqrt{20}$ and $EF = \sqrt{36 + 16} = \sqrt{52}$, so $\overline{DE}$ is longest. Its midpoint is (4, 0), which is 4 from D and from E but $\sqrt{(2-4)^2 + (4-0)^2} = \sqrt{20}$ from F, and $\sqrt{20} \neq 4$. One failing triangle disproves a for-every claim: the conjecture is **false**, and DEF is a counterexample.

(c) Narrow the collection to the kind of triangle that passed: **in every right triangle, the midpoint of the longest side is the same distance from all three corners.** DEF has no right angle, so it no longer belongs to the claim and cannot break it. The repaired claim is a **new conjecture**: one passing case is not a proof, and it needs its own proof before it is a theorem.

SCORING

1 point: (a) longest side and midpoint (3, 4) found, all three distances 5. 1 point: (b) longest side $\overline{DE}$ and midpoint (4, 0) found. 1 point: (b) distance $\sqrt{20} \neq 4$ to F, conjecture called false. 1 point: (c) a repair limited to right triangles. 1 point: (c) the repair called a new conjecture that still needs a proof.

9. Triangle ABC has $A = (0, 0)$, $B = (6, 0)$ and $C = (3, 4)$. Let P be "triangle ABC is equilateral" and Q be "$AC = BC$". (5 points)

(a) Find AB, AC and BC, and give the truth values of P and Q.

(b) Write $\neg P$ in words.

(c) Build the truth table for $\neg P \wedge \neg Q$, say which row this P and Q fall in, and give the truth value of $\neg P \wedge \neg Q$ for this triangle.

ANSWER

(a) $AB = 6$, $AC = 5$, $BC = 5$; P false, Q true. (b) "Triangle ABC is not equilateral." (c) Column F, F, F, T; row 3; false.

SOLUTION

(a) $AB = 6 - 0 = 6$. $AC = \sqrt{3^2 + 4^2} = 5$ and $BC = \sqrt{(6-3)^2 + (0-4)^2} = 5$. The sides are not all equal, so P is false; $AC = BC$, so Q is true.

(b) "Triangle ABC is not equilateral." (It is true, because P is false.)

(c)

Row	P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
1	T	T	F	F	F
2	T	F	F	T	F
3	F	T	T	F	F
4	F	F	T	T	T

P is false and Q is true, which is row 3. There $\neg Q$ is false, so $\neg P \wedge \neg Q$ is false.

SCORING

1 point for the three lengths; 1 point for P false and Q true; 1 point for the negation in words; 1 point for the table with final column F, F, F, T; 1 point for row 3 and the value false.

10. A triangle is **isosceles** if at least two of its sides are congruent. (5 points)

(a) Use the distance formula to decide whether the triangle with corners $A(1, 1)$, $B(5, 4)$ and $C(8, 0)$ is isosceles.

(b) Let t be any number other than 0. Show that the triangle with corners $D(-3, 0)$, $E(3, 0)$ and $F(0, t)$ is isosceles.

(c) Find the positive value of t for which all three sides of $\triangle DEF$ are congruent. For that value, is $\triangle DEF$ isosceles under the definition above? Would it be isosceles under the other definition some books use, "exactly two of its sides are congruent"?

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ANSWER

(a) Yes: $AB = BC = 5$ and $AC = \sqrt{50}$. (b) $FD = FE = \sqrt{9 + t^2}$. (c) $t = 3\sqrt{3}$; isosceles under "at least two", not isosceles under "exactly two".

SOLUTION

(a) $AB = \sqrt{4^2 + 3^2} = 5$, $BC = \sqrt{3^2 + (-4)^2} = 5$ and $AC = \sqrt{7^2 + (-1)^2} = \sqrt{50}$. Two sides, $\overline{AB}$ and $\overline{BC}$, have the same length, so they are congruent, and by the definition the triangle is isosceles.

(b) Since $t \neq 0$, the point F is not on the x-axis, so D, E, F are not collinear and they make a triangle. By the distance formula,

$$FD = \sqrt{(0 - (-3))^2 + (t - 0)^2} = \sqrt{9 + t^2}$$

$$FE = \sqrt{(0 - 3)^2 + (t - 0)^2} = \sqrt{9 + t^2}$$

So $FD = FE$, two sides are congruent, and the triangle meets the definition: it is isosceles, whatever the value of t.

(c) $DE = 3 - (-3) = 6$, and by (b) the other two sides are already equal, so all three are congruent exactly when $\sqrt{9 + t^2} = 6$. Squaring, $9 + t^2 = 36$, so $t^2 = 27$ and the positive value is $t = \sqrt{27} = 3\sqrt{3}$.

Under the definition above the triangle is isosceles: three congruent sides include at least two. Under "exactly two sides congruent" it is not, because it has three congruent sides, not exactly two. Neither verdict is a mistake. The two definitions are different decisions about one word, so a careful writer says which one is in force.

SCORING

1 point: (a) three lengths and the verdict isosceles. 2 points: (b) both distances as $\sqrt{9 + t^2}$ (1), and the fold to "isosceles" for every $t \neq 0$ (1). 1 point: (c) $DE = 6$ and $t = 3\sqrt{3}$. 1 point: (c) isosceles under "at least two", not under "exactly two", each with its reason.

11. Rectangle $ABCD$ has $A(0, 0)$, $B(8, 0)$, $C(8, 6)$ and $D(0, 6)$. For a point P, compare $PA^2 + PC^2$ with $PB^2 + PD^2$. (5 points)

(a) Find both sums for $P(2, 1)$ and for $P(5, 4)$.

(b) State a conjecture about every rectangle $ABCD$ and every point P inside it.

(c) Prove your conjecture. Place the rectangle as $A(0, 0)$, $B(w, 0)$, $C(w, h)$, $D(0, h)$, and let $P = (x, y)$.

(d) Does the same equation hold in every parallelogram? Test the parallelogram $A(0, 0)$, $B(4, 0)$, $C(6, 2)$, $D(2, 2)$, with P the point where its diagonals cross. Then say which hypothesis your proof in (c) used.

ANSWER

(a) For $P(2, 1)$ both sums are 66; for $P(5, 4)$ both are 54. (b) For every rectangle $ABCD$ and every point P inside it, $PA^2 + PC^2 = PB^2 + PD^2$. (c) Both sums equal $x^2 + y^2 + (x - w)^2 + (y - h)^2$. (d) No: with $P(3, 1)$ the sums are 20 and 4. The proof used the right angles of the rectangle, in placing its corners.

SOLUTION

(a) For $P(2, 1)$, by the distance formula: $PA^2 = 4 + 1 = 5$, $PC^2 = 36 + 25 = 61$, sum 66; $PB^2 = 36 + 1 = 37$, $PD^2 = 4 + 25 = 29$, sum 66.

For $P(5, 4)$: $PA^2 = 25 + 16 = 41$, $PC^2 = 9 + 4 = 13$, sum 54; $PB^2 = 9 + 16 = 25$, $PD^2 = 25 + 4 = 29$, sum 54.

(b) Conjecture: **for every rectangle $ABCD$ and every point P inside it,** $PA^2 + PC^2 = PB^2 + PD^2$.

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(c) By the distance formula,

$$PA^2 + PC^2 = \left(x^2 + y^2\right) + \left((x-w)^2 + (y-h)^2\right)$$
$$PB^2 + PD^2 = \left((x-w)^2 + y^2\right) + \left(x^2 + (y-h)^2\right)$$

Both are $x^2 + y^2 + (x-w)^2 + (y-h)^2$, so they are equal. Every rectangle can be moved into this position, and the letters cover every w, h, x and y at once, so this is a proof. It never used that P is inside, so the equation holds for every point P of the plane.

(d) The diagonals cross at their common midpoint: $\overline{AC}$ and $\overline{BD}$ both have midpoint (3, 1), so $P = (3, 1)$. Then $PA^2 + PC^2 = (9 + 1) + (9 + 1) = 20$ and $PB^2 + PD^2 = (1 + 1) + (1 + 1) = 4$. They differ, so **the equation fails** for this parallelogram, and the claim cannot be widened to every parallelogram.

The proof in (c) used the right angles: they are what allowed B on the x-axis, D on the y-axis and C at (w, h). A proof that did not use them would also prove the false claim about parallelograms.

SCORING

(a) 1 point: all four sums correct. (b) 1 point: a for-every statement naming the rectangle, the point and the equation. (c) 2 points: 1 for writing all four squared distances with letters; 1 for showing the sums are equal and saying the letters cover every case. (d) 1 point: the crossing point (3, 1), the sums 20 and 4, and the right angles (rectangle) named as the hypothesis used.

12. Points A, B, C and D lie on a line in that order, and $AB = CD$. M is the midpoint of $\overline{BC}$. (5 points)

(a) Take $A(-3, 1)$, $B(1, 3)$, $C(5, 5)$ and $D(9, 7)$. Check that $AB = CD$, find M, and check that M is also the midpoint of $\overline{AD}$.

(b) Prove that M is the midpoint of $\overline{AD}$ for every four points that fit the first sentence. Write a two-column proof.

(c) Explain why your work in (a) alone does not prove (b).

ANSWER

(a) $AB = CD = \sqrt{20}$; $M = (3, 4)$, and the midpoint of $\overline{AD}$ is also (3, 4). (b) See the proof. (c) It checks one set of four points; (b) is about every set.

SOLUTION

(a) $AB = \sqrt{4^2 + 2^2} = \sqrt{20}$ and $CD = \sqrt{4^2 + 2^2} = \sqrt{20}$. The midpoint of $\overline{BC}$ is $M = \left(\frac{1+5}{2}, \frac{3+5}{2}\right) = (3, 4)$. The midpoint of $\overline{AD}$ is $\left(\frac{-3+9}{2}, \frac{1+7}{2}\right) = (3, 4)$, the same point.

(b) M is between B and C, so the order on the line is A, B, M, C, D.

Statements	Reasons
1. $AB = CD$	given
2. $BM = MC$	definition of midpoint: M is the midpoint of $\overline{BC}$
3. $AM = AB + BM$	segment addition postulate: B is between A and M
4. $MD = MC + CD$	segment addition postulate: C is between M and D
5. $MD = BM + AB$	substitution: lines 1 and 2 into line 4
6. $AM = MD$	substitution: lines 3 and 5 (both equal $AB + BM$)
7. M is the midpoint of $\overline{AD}$	definition of midpoint: M is between A and D, and line 6

(c) Part (a) checks one set of four points. The claim is about every set of four points on a line with $AB = CD$, and there are endlessly many; one case that works is evidence, not a proof. The deduction in (b) uses only the givens, so it covers them all.

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SCORING

(a) 1 point: both lengths and both midpoints correct. (b) 3 points: 1 for the two segment addition lines with the betweenness that justifies them, 1 for the substitutions reaching $AM = MD$, 1 for concluding the midpoint by its definition with M between A and D. (c) 1 point: one case cannot cover every case.

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