Geometry Unit 1 worksheet: Biconditional (answer key)

ANSWERS AT A GLANCE

1. (a) T, T, row 1, true (b) T, F, row 2, false (c) F, T, row 3, false (d) F, F, row 4, true (e) F, T, row 3, false (f) T, T, row 1, true
2. (a) (i) A, (ii) B, (iii) A, (iv) B, (v) B, (vi) A. (b) (A) true, (B) false; for example $M = (2, 3)$. (c) No.
3. (a) $k = -3$ and $k = -2$, both on row 2. (b) For example "$k > 1$ or $k < -1$" (the same as $|k| \geq 2$).
4. (a) T, F, F, T: equivalent. (b) T, T, T, T: not equivalent, differs on rows 2 and 3. (c) T, F, F, T: equivalent. (d) T, F, T, T: not equivalent, differs on row 3.
5. (a) sufficient, not necessary (b) necessary and sufficient (c) sufficient, not necessary (d) neither (e) necessary, not sufficient
6. $PA < PB \iff k^2 + 16 < k^2 - 12k + 40 \iff 12k < 24 \iff k < 2$

WORKED SOLUTIONS

1. Each statement has the form $P \leftrightarrow Q$, with P the part before "if and only if". Decide P and Q, name the row of the table it falls on, and give its truth value.

(a) The point (3, 5) lies on the line $y = 2x - 1$ if and only if the slope from (0, 0) to (1, 2) is 2.

(b) The segment from (0, 0) to (6, 8) has length 10 if and only if its midpoint is (4, 3).

(c) An angle of 95° is acute if and only if 95 + 85 = 180.

(d) The points (0, 0), (1, 2) and (3, 5) are collinear if and only if a triangle can have two obtuse angles.

(e) The segment from (−2, 1) to (4, 9) has length 14 if and only if its midpoint is (1, 5).

(f) The point (2, 7) lies on the line $y = 3x + 1$ if and only if it lies on the line $y = -x + 9$.

ANSWER

(a) T, T, row 1, true (b) T, F, row 2, false (c) F, T, row 3, false (d) F, F, row 4, true (e) F, T, row 3, false (f) T, T, row 1, true

SOLUTION

(a) 2(3) − 1 = 5, so P is true. The slope is $\dfrac{2-0}{1-0} = 2$, so Q is true. Row 1: **true**.

(b) The length is $\sqrt{6^2 + 8^2} = \sqrt{100} = 10$, so P is true. The midpoint is $\left(\dfrac{6}{2}, \dfrac{8}{2}\right) = (3, 4)$, not (4, 3), so Q is false. Row 2: **false**.

(c) An acute angle measures less than 90°, so P is false. 95 + 85 = 180 is true. Row 3: **false**.

(d) The slope from (0, 0) to (1, 2) is 2 and from (1, 2) to (3, 5) is $\dfrac{3}{2}$. The slopes differ, so P is false. Two obtuse angles already add to more than 180°, so Q is false. Row 4: **true**, because the parts agree.

(e) The length is $\sqrt{6^2 + 8^2} = 10$, not 14, so P is false. The midpoint is $\left(\dfrac{-2+4}{2}, \dfrac{1+9}{2}\right) = (1, 5)$, so Q is true. Row 3: **false**.

(f) 3(2) + 1 = 7 and −2 + 9 = 7, so both parts are true. Row 1: **true**.

2. A, B and M are points in a plane. Here are two conditionals.

(A) If M is the midpoint of $\overline{AB}$, then $AM = MB$.

(B) If $AM = MB$, then M is the midpoint of $\overline{AB}$.

(a) Each sentence below says the same thing as (A) or as (B). Say which.

(i) M is the midpoint of $\overline{AB}$ only if $AM = MB$. (ii) $AM = MB$ is sufficient for M to be the midpoint of $\overline{AB}$. (iii) $AM = MB$ if M is the midpoint of $\overline{AB}$. (iv) Being the midpoint of $\overline{AB}$ is necessary for M to have $AM =$

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MB. (v) $AM = MB$ only if M is the midpoint of $\overline{AB}$. (vi) $AM = MB$ is necessary for M to be the midpoint of $\overline{AB}$.

(b) Decide whether (A) and (B) are true. For one that is false, take $A = (0, 0)$ and $B = (4, 0)$ and give a point M that shows it.

(c) Is "M is the midpoint of $\overline{AB}$ if and only if $AM = MB$" true for all points?

ANSWER

(a) (i) A, (ii) B, (iii) A, (iv) B, (v) B, (vi) A. (b) (A) true, (B) false; for example $M = (2, 3)$. (c) No.

SOLUTION

(a) Put the sufficient condition, or the part after "if", into the "if" slot, and the necessary condition, or the part after "only if", into the "then" slot.

(i) "Only if" introduces the necessary condition: if M is the midpoint, then $AM = MB$. **(A)**. (ii) The sufficient condition goes after "if": if $AM = MB$, then M is the midpoint. **(B)**. (iii) The part after "if" is the hypothesis: if M is the midpoint, then $AM = MB$. **(A)**. (iv) The necessary condition goes after "then": if $AM = MB$, then M is the midpoint. **(B)**. (v) If $AM = MB$, then M is the midpoint. **(B)**. (vi) If M is the midpoint, then $AM = MB$. **(A)**.

(b) (A) is true: a midpoint splits the segment into two equal parts. (B) is false. Take $M = (2, 3)$: $AM = \sqrt{2^2 + 3^2} = \sqrt{13}$ and $MB = \sqrt{(4-2)^2 + 3^2} = \sqrt{13}$, so $AM = MB$, but the midpoint of $\overline{AB}$ is $(2, 0)$, and M is not even on the segment. (Any point $(2, y)$ with $y \neq 0$ works.)

(c) No. The biconditional needs both (A) and (B), and (B) fails at $M = (2, 3)$. Being the midpoint is sufficient for $AM = MB$ but not necessary.

3. For an integer k from −3 to 3, let S be the statement

"The point (k, k^2) lies above the line $y = 2$ if and only if $k > 1$."

(a) For which k is S false? Name the row of the table each one falls on.

(b) Replace "$k > 1$" by a different condition on k so that the new statement is true for every integer k.

ANSWER

(a) $k = -3$ and $k = -2$, both on row 2. (b) For example "$k > 1$ or $k < -1$" (the same as $|k| \geq 2$).

SOLUTION

(a) The point lies above $y = 2$ when $k^2 > 2$. For $k = -3, -2, -1, 0, 1, 2, 3$ the squares are 9, 4, 1, 0, 1, 4, 9, so the first part is T, T, F, F, F, T, T. The second part, $k > 1$, is F, F, F, F, F, T, T.

The parts disagree only at $k = -3$ and $k = -2$. There the first part is true and the second false, which is **row 2**, so S is false at $k = -3$ and $k = -2$. On the other five values the parts agree and S is true.

(b) The second part has to hold exactly when $k^2 > 2$. For an integer that means $k \geq 2$ or $k \leq -2$, so "$k > 1$ or $k < -1$" works (as does $|k| \geq 2$). Now both parts are true for $|k| \geq 2$ and both false for $k = -1, 0, 1$, so they always agree.

4. Fill in the column of each statement. Decide which ones are logically equivalent to $P \leftrightarrow Q$. For each one that is not, name a row where it differs from $P \leftrightarrow Q$.

(a) $(P \vee Q) \rightarrow (P \wedge Q)$ (b) $(P \rightarrow Q) \vee (Q \rightarrow P)$ (c) $(P \wedge Q) \vee \neg(P \vee Q)$ (d) $P \leftrightarrow (P \wedge Q)$

ANSWER

(a) T, F, F, T: equivalent. (b) T, T, T, T: not equivalent, differs on rows 2 and 3. (c) T, F, F, T: equivalent. (d) T, F, T, T: not equivalent, differs on row 3.

SOLUTION

$P \leftrightarrow Q$ reads T, F, F, T. Rows run P, Q = TT, TF, FT, FF.

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(a) $P \vee Q$ is T, T, T, F and $P \wedge Q$ is T, F, F, F. The conditional fails only where the first is T and the second F, which is rows 2 and 3. Column T, F, F, T: **equivalent**.

(b) $P \to Q$ is T, F, T, T and $Q \to P$ is T, T, F, T. Their "or" is T on every row, so the column is T, T, T, T. **Not equivalent**: it differs on rows 2 and 3.

(c) $P \wedge Q$ is T, F, F, F. $\neg(P \vee Q)$ flips T, T, T, F to F, F, F, T. Their "or" is T, F, F, T: **equivalent**. (It is true exactly when both are true or both are false.)

(d) Compare P (T, T, F, F) with $P \wedge Q$ (T, F, F, F) row by row: match, differ, match, match. Column T, F, T, T. **Not equivalent**: it differs on row 3. (It is the column of $P \to Q$.)

5. Let $A = (0, 0)$ and $B = (6, 0)$, and let $P = (x, y)$ be any point of the plane. For each condition, decide whether it is sufficient but not necessary, necessary but not sufficient, necessary and sufficient, or neither, for $PA = PB$. Give a point that shows each "not".

(a) $P = (3, 5)$ (b) $x = 3$ (c) P is the midpoint of $\overline{AB}$ (d) $y = 0$ (e) $x > 0$

ANSWER

(a) sufficient, not necessary (b) necessary and sufficient (c) sufficient, not necessary (d) neither (e) necessary, not sufficient

SOLUTION

Square both distances: $PA^2 = x^2 + y^2$ and $PB^2 = (x - 6)^2 + y^2 = x^2 - 12x + 36 + y^2$. Distances are not negative, so

$$PA = PB \iff x^2 + y^2 = x^2 - 12x + 36 + y^2 \iff 12x = 36 \iff x = 3.$$

(a) At (3, 5), $PA = \sqrt{34} = PB$, so it is sufficient. The point (3, 0) has $PA = PB = 3$ and is not (3, 5), so it is not necessary. **Sufficient but not necessary.**

(b) The chain above shows $PA = PB$ holds for exactly the points with $x = 3$. **Necessary and sufficient.**

(c) The midpoint (3, 0) has $PA = PB = 3$, so it is sufficient. The point (3, 4) has $PA = PB = 5$ and is not the midpoint, so it is not necessary. **Sufficient but not necessary.**

(d) (0, 0) has $y = 0$ but $PA = 0$ and $PB = 6$: not sufficient. (3, 4) has $PA = PB$ but $y \neq 0$: not necessary. **Neither.**

(e) If $PA = PB$, then $x = 3 > 0$, so it is necessary. (1, 0) has $x > 0$ but $PA = 1$ and $PB = 5$, so it is not sufficient. **Necessary but not sufficient.**

6. Let $A = (0, 0)$ and $B = (6, 2)$. For a real number k, let $P = (k, 4)$. Show that P is closer to A than to B if and only if $k < 2$. Write your work as a chain of $\iff$ steps, and say why each step can be undone.

ANSWER

$PA < PB \iff k^2 + 16 < k^2 - 12k + 40 \iff 12k < 24 \iff k < 2$

SOLUTION

$PA^2 = k^2 + 4^2 = k^2 + 16$ and $PB^2 = (k - 6)^2 + (4 - 2)^2 = k^2 - 12k + 40$.

$$PA < PB \iff PA^2 < PB^2 \iff k^2 + 16 < k^2 - 12k + 40 \iff 12k < 24 \iff k < 2$$

Why each link runs both ways:

- $PA < PB \iff PA^2 < PB^2$: both distances are 0 or more, and for numbers that are 0 or more, the larger one has the larger square and the larger square comes from the larger number.
- The second link only rewrites each square by the distance formula.

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- $k^2 + 16 < k^2 - 12k + 40 \iff 12k < 24$: add $12k - k^2 - 16$ to both sides; subtracting it back undoes this.
- $12k < 24 \iff k < 2$: divide by the positive number 12; multiplying by 12 undoes it, and neither step flips the inequality.

Every link is a biconditional, so the first and last statements hold for exactly the same k. (At $k = 2$ the point is $(2, 4)$ and $PA = PB = \sqrt{20}$, the boundary.)

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