Geometry Unit 1 worksheet: Conditionals, converse, inverse, contrapositive (answer key)

ANSWERS AT A GLANCE

1. (a) If a climber on the wall is under 12, then an adult is with them. Hypothesis: the climber is under 12. Conclusion: an adult is with them. (b) Broken only by (ii) and (v); (i), (iii) and (iv) keep it.
2. (a) $P \to Q$, true (b) $P \to Q$, true (c) $Q \to P$, false (d) $Q \to P$, false (e) $P \to Q$, true (f) $Q \to P$, false. A counterexample to each false one: $m\angle A = 85^\circ$.
3. (a) Diego swapped the parts but did not negate them: he wrote the converse. Contrapositive: if $AM \neq MB$, then M is not the midpoint of $\overline{AB}$. Diego's sentence is false. (b) The negation of an "and" is an "or". Contrapositive: if a point is not $(0, 0)$, then it does not lie on the x-axis or it does not lie on the y-axis. Maya's sentence is false. (c) The negation of "less than $90°$" is "$90°$ or more", not "greater than $90°$". Contrapositive: if an angle's measure is $90°$ or more, then it is not acute. Mateo's sentence is true, but it is not the contrapositive.
4. (a) $k = 6, 7, 8$ (b) $P \to Q$ is false for $k = 2, 4$; the converse is false for $k = 7$. (c) The inverse is false for $k = 7$; the contrapositive is false for $k = 2, 4$. (d) 5 values ($k = 1, 3, 5, 6, 8$).
5. (a) True. (b) Converse: if $PA = PB$, then P lies on the line $y = 2x - 3$. True. (c) Inverse: if P does not lie on $y = 2x - 3$, then $PA \neq PB$. Contrapositive: if $PA \neq PB$, then P does not lie on $y = 2x - 3$. Both true.
6. (a) If every side of a rectangle is 5 cm or shorter, then its perimeter is 20 cm or less. (b) Width $w \leq 5$ and height $h \leq 5$ give perimeter $2w + 2h \leq 10 + 10 = 20$; a statement and its contrapositive are logically equivalent. (c) Converse: if at least one side of a rectangle is longer than 5 cm, then its perimeter is more than 20 cm. False: a 6 cm by 1 cm rectangle has perimeter 14 cm.

WORKED SOLUTIONS

1. A climbing gym posts this rule: "A climber on the wall is under 12 only if an adult is with them."

(a) Write the rule as an if-then statement, and name its hypothesis and its conclusion.

(b) For each climber, say whether the rule is kept or broken, and why.

(i) A 9-year-old climbs with her father. (ii) A 10-year-old climbs alone. (iii) A 14-year-old climbs alone. (iv) A 40-year-old climbs with a friend his own age. (v) An 11-year-old climbs with his 16-year-old sister and no adult.

> **ANSWER**
> (a) If a climber on the wall is under 12, then an adult is with them. Hypothesis: the climber is under 12. Conclusion: an adult is with them. (b) Broken only by (ii) and (v); (i), (iii) and (iv) keep it.
>
> **SOLUTION**
> (a) "Only if" introduces what the first part cannot happen without, so it is the conclusion: *if a climber on the wall is under 12, then an adult is with them*. Hypothesis: the climber is under 12. Conclusion: an adult is with the climber.
>
> (b) The rule is broken only when the hypothesis is true and the conclusion is false: a climber under 12 with no adult.
>
> (i) Under 12, adult present: hypothesis true, conclusion true. Kept.
>
> (ii) Under 12, no adult: hypothesis true, conclusion false. Broken.
>
> (iii) 14 is not under 12, so the hypothesis is false and the rule asks nothing of this climber. Kept (vacuously).
>
> (iv) Not under 12: hypothesis false. Kept (vacuously); the rule does not care who an adult climbs with.
>
> (v) Under 12, and a 16-year-old is not an adult, so no adult is present: hypothesis true, conclusion false. Broken.

2. For an angle $\angle A$, let P be "$m\angle A = 90^\circ$" and let Q be "$m\angle A > 80^\circ$". Say whether each sentence means $P \to Q$ or $Q \to P$. Then say which sentences are true for every angle $\angle A$, and give one angle measure that is a

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counterexample to each false one.

(a) $m\angle A > 80^\circ$ whenever $m\angle A = 90^\circ$.

(b) $m\angle A = 90^\circ$ only if $m\angle A > 80^\circ$.

(c) $m\angle A > 80^\circ$ only if $m\angle A = 90^\circ$.

(d) $m\angle A = 90^\circ$ if $m\angle A > 80^\circ$.

(e) For $m\angle A > 80^\circ$, it is sufficient that $m\angle A = 90^\circ$.

(f) For $m\angle A > 80^\circ$, it is necessary that $m\angle A = 90^\circ$.

ANSWER

(a) $P \to Q$, true (b) $P \to Q$, true (c) $Q \to P$, false (d) $Q \to P$, false (e) $P \to Q$, true (f) $Q \to P$, false. A counterexample to each false one: $m\angle A = 85^\circ$.

SOLUTION

(a) "Whenever" introduces the hypothesis: if $m\angle A = 90^\circ$, then $m\angle A > 80^\circ$. That is $P \to Q$.

(b) "Only if" introduces the conclusion: if $m\angle A = 90^\circ$, then $m\angle A > 80^\circ$. That is $P \to Q$.

(c) If $m\angle A > 80^\circ$, then $m\angle A = 90^\circ$. That is $Q \to P$.

(d) "If" introduces the hypothesis, even though it comes second: if $m\angle A > 80^\circ$, then $m\angle A = 90^\circ$. That is $Q \to P$.

(e) A sufficient condition is the hypothesis: if $m\angle A = 90^\circ$, then $m\angle A > 80^\circ$. That is $P \to Q$.

(f) A necessary condition is the conclusion: if $m\angle A > 80^\circ$, then $m\angle A = 90^\circ$. That is $Q \to P$.

$P \to Q$ is true for every angle, since 90 > 80. $Q \to P$ is false: an angle with $m\angle A = 85^\circ$ has 85 > 80 but 85 ≠ 90, so its hypothesis is true and its conclusion false. So (a), (b) and (e) are true, and (c), (d) and (f) are false, each broken by $m\angle A = 85^\circ$ (any measure above 80° other than 90° works).

3. Each student was asked for the contrapositive of a statement. Find the mistake, write the correct contrapositive, and say whether the student's own sentence is true.

(a) Statement: "If M is the midpoint of $\overline{AB}$, then $AM = MB$." Diego wrote: "If $AM = MB$, then M is the midpoint of $\overline{AB}$."

(b) Statement: "If a point lies on the x-axis and on the y-axis, then it is (0, 0)." Maya wrote: "If a point is not (0, 0), then it does not lie on the x-axis and it does not lie on the y-axis."

(c) Statement: "If an angle is acute, then its measure is less than 90°." Mateo wrote: "If an angle's measure is greater than 90°, then it is not acute."

ANSWER

(a) Diego swapped the parts but did not negate them: he wrote the converse. Contrapositive: if $AM \neq MB$, then M is not the midpoint of $\overline{AB}$. Diego's sentence is false. (b) The negation of an "and" is an "or". Contrapositive: if a point is not (0, 0), then it does not lie on the x-axis or it does not lie on the y-axis. Maya's sentence is false. (c) The negation of "less than 90°" is "90° or more", not "greater than 90°". Contrapositive: if an angle's measure is 90° or more, then it is not acute. Mateo's sentence is true, but it is not the contrapositive.

SOLUTION

(a) Diego swapped hypothesis and conclusion without negating either, which gives the converse. The contrapositive swaps and negates both: *if $AM \neq MB$, then M is not the midpoint of $\overline{AB}$*. Diego's sentence is false: take $A = (0, 0)$, $B = (4, 0)$ and $M = (2, 3)$. Then $AM = \sqrt{4 + 9} = \sqrt{13}$ and $MB = \sqrt{4 + 9} = \sqrt{13}$, so $AM = MB$, but M is not on $\overline{AB}$, so it is not the midpoint.

(b) The conclusion's negation is right, but the hypothesis "on the x-axis and on the y-axis" is negated wrongly: it is false as soon as either part fails, so its negation is "not on the x-axis, or not on the y-axis". Contrapositive: *if a point is not (0, 0), then it does not lie on the x-axis or it does not lie on the y-axis*. Maya's sentence is false: the point (5, 0) is not (0, 0), yet it lies on the x-axis.

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(c) The negation of "the measure is less than 90°" is "the measure is not less than 90°", that is, 90° or more. Mateo left out 90° itself. Contrapositive: *if an angle's measure is 90° or more, then it is not acute*. Mateo's sentence is true (an angle over 90° is obtuse, not acute), but it is a weaker statement: it says nothing about an angle of exactly 90°, so it is not the contrapositive.

4. For each whole number k from 1 to 8, let P be "k is even" and let Q be "a square with sides of k cm has an area of more than 30 cm^2".

(a) For which values of k is Q true?

(b) For which values of k is $P \to Q$ false? For which is its converse false?

(c) For which values of k is the inverse false? For which is the contrapositive false?

(d) For how many of the eight values of k are all four statements true?

ANSWER

(a) $k = 6, 7, 8$ (b) $P \to Q$ is false for $k = 2, 4$; the converse is false for $k = 7$. (c) The inverse is false for $k = 7$; the contrapositive is false for $k = 2, 4$. (d) 5 values ($k = 1, 3, 5, 6, 8$).

SOLUTION

(a) The area is k^2 cm^2. For $k = 1, \ldots, 8$ the areas are 1, 4, 9, 16, 25, 36, 49, 64, and only the last three are more than 30. So Q is true for $k = 6, 7, 8$ and false for $k = 1$ to 5.

(b) $P \to Q$ is false only where P is true and Q is false: k even and $k \le 5$, so $k = 2, 4$. The converse $Q \to P$ is false only where Q is true and P is false: $k \ge 6$ and odd, so $k = 7$.

(c) The inverse $\neg P \to \neg Q$ is false where $\neg P$ is true and $\neg Q$ is false, that is, P false and Q true: $k = 7$, the same case as the converse. The contrapositive $\neg Q \to \neg P$ is false where Q is false and P is true: $k = 2, 4$, the same case as the original.

(d) All four are true exactly where none of the cases above happens, which is where P and Q have the same truth value: both true at $k = 6, 8$, both false at $k = 1, 3, 5$. That is 5 values.

5. Let $A = (4, 0)$ and $B = (0, 2)$. For a point $P(x, y)$, consider the statement: "if P lies on the line $y = 2x - 3$, then $PA = PB$."

(a) Show that the statement is true for every point P.

(b) Write its converse, and decide whether the converse is true for every point P.

(c) Write the inverse and the contrapositive, and say whether each is true.

ANSWER

(a) True. (b) Converse: if $PA = PB$, then P lies on the line $y = 2x - 3$. True. (c) Inverse: if P does not lie on $y = 2x - 3$, then $PA \ne PB$. Contrapositive: if $PA \ne PB$, then P does not lie on $y = 2x - 3$. Both true.

SOLUTION

Work with squared distances: $PA^2 = (x - 4)^2 + y^2$ and $PB^2 = x^2 + (y - 2)^2$, and $PA = PB$ exactly when $PA^2 = PB^2$ (distances are never negative). Expanding,

$$PA^2 - PB^2 = (x^2 - 8x + 16 + y^2) - (x^2 + y^2 - 4y + 4) = -8x + 4y + 12.$$

(a) If $y = 2x - 3$, then $-8x + 4(2x - 3) + 12 = -8x + 8x - 12 + 12 = 0$, so $PA^2 = PB^2$ and $PA = PB$. True for every P on the line.

(b) Converse: *if $PA = PB$, then P lies on the line $y = 2x - 3$*. If $PA = PB$, then $-8x + 4y + 12 = 0$, so $4y = 8x - 12$ and $y = 2x - 3$. The converse is true too.

(c) Inverse: *if P does not lie on $y = 2x - 3$, then $PA \ne PB$*. Contrapositive: *if $PA \ne PB$, then P does not lie on $y = 2x - 3$*. The contrapositive is equivalent to the original, which is true by (a), and the inverse is

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equivalent to the converse, which is true by (b). So all four statements are true. (The line is the perpendicular bisector of $\overline{AB}$.)

6. Claim: "If a rectangle has a perimeter of more than 20 cm, then at least one of its sides is longer than 5 cm."

(a) Write the contrapositive of the claim. Take care with the negation of "at least one of its sides is longer than 5 cm".

(b) Prove the contrapositive, and say why this proves the claim.

(c) Write the converse, and decide whether it is true.

ANSWER

(a) If every side of a rectangle is 5 cm or shorter, then its perimeter is 20 cm or less. (b) Width $w \leq 5$ and height $h \leq 5$ give perimeter $2w + 2h \leq 10 + 10 = 20$; a statement and its contrapositive are logically equivalent. (c) Converse: if at least one side of a rectangle is longer than 5 cm, then its perimeter is more than 20 cm. False: a 6 cm by 1 cm rectangle has perimeter 14 cm.

SOLUTION

(a) "At least one side is longer than 5 cm" is false exactly when no side is longer than 5 cm, that is, every side is 5 cm or shorter. The negation of "perimeter more than 20 cm" is "perimeter 20 cm or less". Contrapositive: *if every side of a rectangle is 5 cm or shorter, then its perimeter is 20 cm or less.*

(b) Let the rectangle have width w and height h, with $w \leq 5$ and $h \leq 5$. Its perimeter is $2w + 2h \leq 2(5) + 2(5) = 20$, so the perimeter is 20 cm or less. The contrapositive holds for every rectangle, and a statement and its contrapositive are logically equivalent, so the claim holds for every rectangle.

(c) Converse: *if at least one side of a rectangle is longer than 5 cm, then its perimeter is more than 20 cm.* False: a rectangle 6 cm by 1 cm has a side longer than 5 cm, but its perimeter is 2(6) + 2(1) = 14 cm.

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