Geometry Unit 1 worksheet: Conjecture, and why a pattern is not a proof (answer key)

ANSWERS AT A GLANCE

1. (a) No change: C is still open. (b) C becomes a theorem. (c) C becomes a false conjecture. (d) No change: still open. (e) No change: still open.
2. (a) $4, 7, 10, 13$; conjecture: a row of n squares uses $3n + 1$ toothpicks. (b) See the solution. (c) 33 squares.
3. (a) Claim A is true: the lengths are $\sqrt{65}, \sqrt{85}, \sqrt{109}, \sqrt{137}$, none of $65, 85, 109, 137$ is a perfect square, and these are all its cases. (b) Nothing: Claim B is still open after (a). (c) Claim B is false: $OP_5 = \sqrt{25 + 144} = 13$.
4. (a) A prime that is at least 3 is odd. (b) $3, 7, 13, 21$; $21 = 3 \cdot 7$, so A is false. (c) Nothing: disproving the stronger A does not disprove the weaker B. (d) B is true (a theorem): $n^2 + n = n(n + 1)$ is even, so $n^2 + n + 1$ is odd.
5. (a) $60°, 90°, 108°, 120°$. (b) False: at $n = 7$ the angle is $\frac{900}{7}$ degrees, not whole. (c) $n = 8, 9, 10, 12$. (d) For every whole number $n \geq 3$ that divides 360, each angle is a whole number of degrees; true because the angle is $180 - \frac{360}{n}$.
6. (a) The last digits of $0^5, 1^5, \ldots, 9^5$ are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$: all ten pass. (b) The last digit of n^5 depends only on the last digit of n, and every whole number ends in one of the ten digits checked.
7. (a) For-every; settles nothing. (b) There-is; proves it. (c) For-every; disproves it. (d) There-is; settles nothing. By the angle sum theorem the claim is false.
8. (a) Every side is 5, but $PR = 6$ and $QS = 8$. (b) Every side is 13, but the diagonals are 10 and 24. (c) No: $PQRS$ has no right angle, so it is outside the new claim. The repair is a new conjecture, not yet a theorem: it needs its own proof.

WORKED SOLUTIONS

1. Conjecture C is a claim about every whole number, and it is open. Each line below is a piece of news about C. Say whether the news changes the status of C, and if it does, say what C becomes.

(a) A computer checks C for ten billion more whole numbers, and every one of them passes.

(b) An argument is published, and checked by experts, that shows C holds for every whole number.

(c) One whole number is found for which C fails, and the failure is checked by hand.

(d) A famous mathematician says in an interview that she is sure C is true.

(e) A magazine starts calling it "the C theorem".

ANSWER

(a) No change: C is still open. (b) C becomes a theorem. (c) C becomes a false conjecture. (d) No change: still open. (e) No change: still open.

SOLUTION

Only a proof or a counterexample changes the status of a conjecture.

(a) Ten billion passing cases are finitely many, and C is about every whole number. The unchecked numbers are still there, so C is still **open**.

(b) An argument that covers every case is a proof, so C is now a **theorem**.

(c) C is a for-every claim, so one failing case disproves it. C is now a **false conjecture**.

(d) Belief, even an expert's, is not a proof. C is still **open**.

(e) A name does not set the status. Calling it a theorem proves nothing, so C is still **open**.

2. A row of n squares is built from toothpicks. Each side of a square is one toothpick, and two neighboring squares share the toothpick between them.

(a) Count the toothpicks for rows of 1, 2, 3 and 4 squares, and state a conjecture for a row of n squares.

(b) Give a reason that covers every n, with n kept as a letter.

(c) A row is built from exactly 100 toothpicks. How many squares does it have?

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ANSWER

(a) 4, 7, 10, 13; conjecture: a row of n squares uses $3n + 1$ toothpicks. (b) See the solution. (c) 33 squares.

SOLUTION

(a) The counts are 4, 7, 10, 13. Each is 3 more than the one before. Conjecture: **for every whole number $n \geq 1$, a row of n squares uses $3n + 1$ toothpicks.**

(b) The first square needs 4 toothpicks. Each square added to the right end shares its left side with the square already there, so it needs only 3 new toothpicks: top, bottom and right. A row of n squares is the first square and $n - 1$ added ones, so it uses

$$4 + 3(n - 1) = 3n + 1$$

toothpicks. Nothing in the argument depends on which n was chosen, so it covers every n and the conjecture is a theorem.

(c) $3n + 1 = 100$ gives $3n = 99$, so $n = 33$ squares.

3. O is the origin (0, 0). For each whole number $n \geq 1$, P_n is the point $(n, n + 7)$ on the line $y = x + 7$.

(a) Claim A: for every whole number n from 1 to 4, the length OP_n is not a whole number. Settle Claim A.

(b) Claim B: for every whole number $n \geq 1$, the length OP_n is not a whole number. What do your checks in (a) show about Claim B?

(c) Settle Claim B.

ANSWER

(a) Claim A is true: the lengths are $\sqrt{65}$, $\sqrt{85}$, $\sqrt{109}$, $\sqrt{137}$, none of 65, 85, 109, 137 is a perfect square, and these are all its cases. (b) Nothing: Claim B is still open after (a). (c) Claim B is false: $OP_5 = \sqrt{25 + 144} = 13$.

SOLUTION

(a) By the distance formula, $OP_n = \sqrt{n^2 + (n + 7)^2}$. Claim A has only four cases: $OP_1 = \sqrt{1 + 64} = \sqrt{65}$, $OP_2 = \sqrt{4 + 81} = \sqrt{85}$, $OP_3 = \sqrt{9 + 100} = \sqrt{109}$, $OP_4 = \sqrt{16 + 121} = \sqrt{137}$. None is whole, because $64 < 65 < 81$, $81 < 85 < 100$, $100 < 109 < 121$ and $121 < 137 < 144$: each number sits strictly between two neighboring squares. Going through every case of a finite claim settles it, so Claim A is **true**.

(b) Claim B is about every whole number $n \geq 1$, infinitely many cases. The four checks in (a) cover four of them and leave the rest unchecked, so they **settle nothing** about B.

(c) Try the next case: $OP_5 = \sqrt{25 + 144} = \sqrt{169} = 13$, a whole number. ($OP_5$ is the long side of a right triangle with legs 5 and 12.) One failing case disproves a for-every claim, so Claim B is **false**, with $n = 5$ as a counterexample.

4. Conjecture A: for every whole number $n \geq 1$, the number $n^2 + n + 1$ is prime.

Conjecture B: for every whole number $n \geq 1$, the number $n^2 + n + 1$ is odd.

(a) Explain why A implies B. (Note that $n^2 + n + 1 \geq 3$ for every $n \geq 1$.)

(b) Test A for $n = 1, 2, 3, 4$ and settle A.

(c) After (b), what do you know about B? Explain.

(d) Settle B.

ANSWER

(a) A prime that is at least 3 is odd. (b) 3, 7, 13, 21; $21 = 3 \cdot 7$, so A is false. (c) Nothing: disproving the stronger A does not disprove the weaker B. (d) B is true (a theorem): $n^2 + n = n(n + 1)$ is even, so $n^2 + n + 1$ is odd.

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SOLUTION

(a) The only even prime is 2. If A holds, then each number $n^2 + n + 1$ is a prime that is at least 3, so it is not 2, so it is odd. That is B, so A implies B, and A is the stronger claim.

(b) The values are 3, 7, 13, 21. The first three are prime, but 21 = 3 · 7. One failing case disproves a for-every claim, so A is **false**, with $n = 4$ as a counterexample.

(c) **Nothing.** If A implies B, disproving B would disprove A, but disproving A does not disprove B: B asks for less than A does, and may still hold. B is still open after (b).

(d) $n^2 + n = n(n + 1)$ is the product of two neighboring whole numbers. One of any two neighboring whole numbers is even, so the product is even, and one more than an even number is odd. This argument keeps n as a letter, so it covers every $n \geq 1$: B is a **theorem**.

5. Each angle of a regular polygon with n sides measures $\frac{180(n-2)}{n}$ degrees. Conjecture: for every whole number $n \geq 3$, each angle of a regular polygon with n sides is a whole number of degrees.

(a) Find the angle for $n = 3, 4, 5, 6$.

(b) Settle the conjecture.

(c) Which values of n from 8 to 12 give a whole number of degrees?

(d) Repair the conjecture by narrowing the collection, and prove the repaired claim. (Hint: $\frac{180(n-2)}{n} = 180 - \frac{360}{n}$.)

ANSWER

(a) 60°, 90°, 108°, 120°. (b) False: at $n = 7$ the angle is $\frac{900}{7}$ degrees, not whole. (c) $n = 8, 9, 10, 12$. (d) For every whole number $n \geq 3$ that divides 360, each angle is a whole number of degrees; true because the angle is $180 - \frac{360}{n}$.

SOLUTION

(a) $\frac{180 \cdot 1}{3} = 60$, $\frac{180 \cdot 2}{4} = 90$, $\frac{180 \cdot 3}{5} = 108$, $\frac{180 \cdot 4}{6} = 120$. Four whole numbers.

(b) At $n = 7$ the angle is $\frac{180 \cdot 5}{7} = \frac{900}{7} = 128\frac{4}{7}$ degrees, which is not whole. One failing case disproves a for-every claim, so the conjecture is **false**.

(c) $n = 8$: 135°. $n = 9$: 140°. $n = 10$: 144°. $n = 11$: $\frac{1620}{11}$, not whole. $n = 12$: 150°. So $n = 8, 9, 10, 12$.

(d) The working values 3, 4, 5, 6, 8, 9, 10, 12 all divide 360, and 7 and 11 do not. Repaired claim: **for every whole number $n \geq 3$ that divides 360, each angle of a regular n-sided polygon is a whole number of degrees.** Proof: $\frac{180(n-2)}{n} = \frac{180n - 360}{n} = 180 - \frac{360}{n}$. When n divides 360, $\frac{360}{n}$ is a whole number, so $180 - \frac{360}{n}$ is a whole number. The argument keeps n a letter, so the repaired claim is a theorem.

6. Claim: for every whole number n, the last digit of n^5 is the same as the last digit of n.

(a) Check the claim for $n = 0, 1, 2, \ldots, 9$.

(b) Explain why these ten checks prove the claim for every whole number n, even though the claim has infinitely many cases.

ANSWER

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(a) The last digits of $0^5, 1^5, \ldots, 9^5$ are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9: all ten pass. (b) The last digit of n^5 depends only on the last digit of n, and every whole number ends in one of the ten digits checked.

SOLUTION

(a) $0^5 = 0$, $1^5 = 1$, $2^5 = 32$, $3^5 = 243$, $4^5 = 1024$, $5^5 = 3125$, $6^5 = 7776$, $7^5 = 16807$, $8^5 = 32768$, $9^5 = 59049$. The last digits are 0, 1, 2, ..., 9, each matching n.

(b) Ten checks of an infinite claim settle nothing on their own; they settle this one because of one more fact. The last digit of a product depends only on the last digits of the factors: writing $n = 10a + d$, where d is the last digit, every term of $(10a + d)^5$ except d^5 has a factor 10. So n^5 ends in the same digit as d^5, and by (a) that is d, the last digit of n. Every whole number ends in one of 0, ..., 9, so the ten checks cover every case. The proof is that argument together with the checks, not the checks alone.

7. Each claim is followed by one fact. Say whether the claim is a for-every claim or a there-is claim, and whether the fact proves the claim, disproves it, or settles nothing.

(a) Claim: every triangle with two sides of equal length has two angles of equal measure. Fact: a triangle with sides 5, 5 and 8 is measured, and two of its angles are equal.

(b) Claim: there is a rectangle whose area and perimeter are the same number. Fact: a square with sides of length 4.

(c) Claim: in every triangle with a right angle, the two sides that form the right angle have different lengths. Fact: a triangle with angles 45°, 45° and 90° has two equal sides forming its right angle.

(d) Claim: there is a triangle with two right angles. Fact: a computer draws a million triangles at random, and none of them has two right angles. After answering, settle this claim using the angle sum theorem.

ANSWER

(a) For-every; settles nothing. (b) There-is; proves it. (c) For-every; disproves it. (d) There-is; settles nothing. By the angle sum theorem the claim is false.

SOLUTION

(a) For-every. One triangle that passes is one case, and the claim covers every triangle, so the fact **settles nothing**.

(b) There-is, so one example proves it. A square is a rectangle (four right angles). Its area is $4 \cdot 4 = 16$ and its perimeter is $4 + 4 + 4 + 4 = 16$, the same number. The fact **proves** the claim.

(c) For-every, so one failing case disproves it. The 45°-45°-90° triangle has a right angle whose two sides are equal. The fact **disproves** the claim.

(d) There-is. A failed search covers only the triangles searched, so the fact **settles nothing**. The angle sum theorem settles it: two right angles already add to 90° + 90° = 180°, which leaves 0° for the third angle, and a triangle cannot have an angle of 0°. So no triangle has two right angles, and the claim is **false**. A reason did what no search could.

8. Conjecture: every quadrilateral with four sides of equal length has two diagonals of equal length.

(a) Show that the quadrilateral $PQRS$ with $P(0, 3)$, $Q(4, 0)$, $R(0, -3)$ and $S(-4, 0)$ is a counterexample.

(b) One proposed repair: "every quadrilateral with four sides of equal length, other than $PQRS$, has diagonals of equal length." Show that this repair also fails, using the quadrilateral with corners (0, 5), (12, 0), (0, −5) and (−12, 0).

(c) A second proposed repair: "every quadrilateral with four sides of equal length and at least one right angle has diagonals of equal length." Does $PQRS$ still break this claim? Is the repaired claim a theorem now? Explain.

ANSWER

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(a) Every side is 5, but $PR = 6$ and $QS = 8$. (b) Every side is 13, but the diagonals are 10 and 24. (c) No: $PQRS$ has no right angle, so it is outside the new claim. The repair is a new conjecture, not yet a theorem: it needs its own proof.

SOLUTION

(a) By the distance formula, $PQ = \sqrt{4^2 + 3^2} = 5$, and in the same way $QR = RS = SP = 5$. The diagonals are $PR = 3 - (-3) = 6$ and $QS = 4 - (-4) = 8$. Four equal sides, unequal diagonals: $PQRS$ is a counterexample, and the conjecture is false.

(b) Each side is $\sqrt{12^2 + 5^2} = 13$, and the diagonals are 5 − (−5) = 10 and 12 − (−12) = 24. This shape is not $PQRS$, so it belongs to the "repaired" claim and breaks it. Leaving out the one known counterexample does not remove the reason the claim fails; there are many more.

(c) At Q, the sides run to P and R, along (−4, 3) and (−4, −3). The slopes of $\overline{QP}$ and $\overline{QR}$ are $-\frac{3}{4}$ and $\frac{3}{4}$, whose product is $-\frac{9}{16}$, not −1, so the angle at Q is not right; the same holds at each corner. So $PQRS$ is **not** in the new collection and cannot break it. But the repaired claim is a **new conjecture**: the shapes in its collection have not been examined by this work, and it needs its own proof before it is a theorem.

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