Geometry Unit 1 worksheet: Counterexample (answer key)

ANSWERS AT A GLANCE

1. (a) not a counterexample (b) a counterexample (c) not a counterexample (d) not a counterexample
2. Many answers; for example: (a) $x = 1$ (b) $x = -3$ (c) $a = 1, b = 3$ (d) $x = -1$ (e) $n = 3$ (f) $a = 2, b = 1, c = -1$
3. (a) No: $AB = 3$, $BC = 4$ and $AC = 7$, so the claim holds here. (b) Many answers; for example $A(0,0)$, $B(3,0)$, $C(3,4)$: $AC = 5$. (c) Many answers; for example $A(0,0)$, $B(3,0)$, $C(-1,0)$: $AC = 1$.
4. Many answers; for example: (a) two right angles, 90° + 90° = 180° (b) from (−1, 0) to (1, 0) and from (−2, 0) to (2, 0) (c) two skew edges of a box, such as $\overleftrightarrow{AB}$ and $\overleftrightarrow{CG}$
5. (a) Many answers; for example $\ell = \overleftrightarrow{AB}$, $m = \overleftrightarrow{AE}$, $n = \overleftrightarrow{AD}$: $\ell \perp m$ and $m \perp n$, but ℓ and n meet at A, so they are not parallel. (b) No. A counterexample refutes only the claim as stated, about lines anywhere in space; the narrower claim about three lines in one plane is a different claim and needs its own proof or counterexample.
6. (a) Dividing by x needs $x \neq 0$. (b) $x = 0$: $0^2 = 5 \cdot 0$ is true, but $0 \neq 5$. (c) For every nonzero real number x, if $x^2 = 5x$, then $x = 5$.
7. The row with p true and q false: $\neg(p \wedge q)$ is true and $\neg p \wedge \neg q$ is false. (The row with p false and q true works too.)

WORKED SOLUTIONS

1. Claim: for every point $P(x, y)$ in the coordinate plane, if $x + y = 4$, then the distance from P to the origin O is at most 4. For each point, decide whether it is a counterexample to the claim, and say why.

(a) $P(2, 2)$

(b) $P(5, -1)$

(c) $P(3, 3)$

(d) $P(4, 0)$

ANSWER

(a) not a counterexample (b) a counterexample (c) not a counterexample (d) not a counterexample

SOLUTION

A counterexample must make $x + y = 4$ true and make $OP \leq 4$ false. Compare squares to avoid decimals: $OP \leq 4$ exactly when $OP^2 \leq 16$.

(a) 2 + 2 = 4, so the hypothesis holds. $OP = \sqrt{2^2 + 2^2} = \sqrt{8}$, and 8 ≤ 16, so $OP \leq 4$. The claim holds here: not a counterexample.

(b) 5 + (−1) = 4, so the hypothesis holds. $OP = \sqrt{5^2 + (-1)^2} = \sqrt{26}$, and 26 > 16, so $OP > 4$. Hypothesis true, conclusion false: a counterexample, and the claim is false.

(c) 3 + 3 = 6, so $x + y = 4$ is false. The claim promises nothing about this point, so it cannot break it: not a counterexample.

(d) 4 + 0 = 4, and $OP = \sqrt{4^2 + 0^2} = 4$. "At most 4" allows 4 itself, so the conclusion holds: not a counterexample.

The claim's author pictured the points of the line $x + y = 4$ between (4, 0) and (0, 4). Past those two points, where one coordinate is negative, the line runs farther and farther from O.

2. Each claim is false. Give a counterexample, and show exactly that it meets every hypothesis and makes the conclusion false.

(a) For every real x, $(x + 2)^2 = x^2 + 4$.

(b) For every real x, $\sqrt{x^2} = x$.

(c) For all positive integers a and b, if $a + b$ is even, then a and b are both even.

(d) For every nonzero real x, $x + \dfrac{1}{x} \geq 2$.

(e) For every positive integer n, $3^n > n^3$.

(f) For all real a, b and c, if $ac < bc$, then $a < b$.

ANSWER

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Many answers; for example: (a) $x = 1$ (b) $x = -3$ (c) $a = 1, b = 3$ (d) $x = -1$ (e) $n = 3$ (f) $a = 2, b = 1, c = -1$

SOLUTION

(a) $x = 1$: $(1 + 2)^2 = 9$, but $1^2 + 4 = 5$, and $9 \neq 5$. (Any $x \neq 0$ works: the left side has the extra term $4x$.)

(b) $x = -3$: $\sqrt{(-3)^2} = \sqrt{9} = 3$, and $3 \neq -3$. Any negative x works.

(c) $a = 1$, $b = 3$: both are positive integers and $a + b = 4$ is even, so the hypothesis holds. But 1 is odd, so they are not both even.

(d) $x = -1$ is nonzero, and $-1 + \dfrac{1}{-1} = -2$, which is not ≥ 2. Any negative x works.

(e) $n = 3$: $3^3 = 27$ and $3^3 = 27$, and $27 > 27$ is false. (The claim holds for $n = 1$ and $n = 2$, since $3 > 1$ and $9 > 8$; the two sides are equal at $n = 3$.)

(f) $a = 2$, $b = 1$, $c = -1$: $ac = -2$ and $bc = -1$, and $-2 < -1$, so the hypothesis holds. But $2 < 1$ is false. Dividing $ac < bc$ by c keeps the direction only when c is positive, so a negative c is where to look.

3. Claim: for all points A, B and C in the coordinate plane, if $AB = 3$ and $BC = 4$, then $AC = 7$.

(a) Is $A(0, 0)$, $B(3, 0)$, $C(7, 0)$ a counterexample? Explain.

(b) Find a counterexample in which A, B and C are not on one line, and check it with the distance formula.

(c) Find a counterexample in which A, B and C are all on one line, and check it.

ANSWER

(a) No: $AB = 3$, $BC = 4$ and $AC = 7$, so the claim holds here. (b) Many answers; for example $A(0, 0)$, $B(3, 0)$, $C(3, 4)$: $AC = 5$. (c) Many answers; for example $A(0, 0)$, $B(3, 0)$, $C(-1, 0)$: $AC = 1$.

SOLUTION

A counterexample must make $AB = 3$ and $BC = 4$ true and make $AC = 7$ false.

(a) $AB = |3 - 0| = 3$, $BC = |7 - 3| = 4$ and $AC = |7 - 0| = 7$. Hypothesis true and conclusion true: the claim holds here, so this is not a counterexample. This is the picture the claim's author had in mind, with B between A and C.

(b) Keep $A(0, 0)$ and $B(3, 0)$, and put C straight above B at $C(3, 4)$. Then

$$AB = 3, \quad BC = \sqrt{(3 - 3)^2 + (4 - 0)^2} = 4, \quad AC = \sqrt{(3 - 0)^2 + (4 - 0)^2} = \sqrt{25} = 5$$

The hypothesis holds and $5 \neq 7$, so the claim is false.

(c) Keep $A(0, 0)$ and $B(3, 0)$, and put C on the other side of B, back past A, at $C(-1, 0)$. Then $AB = 3$, $BC = |3 - (-1)| = 4$ and $AC = |-1 - 0| = 1$, and $1 \neq 7$. Even on one line the claim fails, because B need not lie between A and C.

4. Each claim is false. Give a counterexample to each, and show why it breaks the claim.

(a) If two angles are supplementary, then one of them is obtuse.

(b) If two segments have the same midpoint, then they have the same length.

(c) If two different lines in space do not intersect, then they are parallel.

ANSWER

Many answers; for example: (a) two right angles, 90° + 90° = 180° (b) from (−1, 0) to (1, 0) and from (−2, 0) to (2, 0) (c) two skew edges of a box, such as $\overleftrightarrow{AB}$ and $\overleftrightarrow{CG}$

SOLUTION

(a) Two right angles: 90° + 90° = 180°, so they are supplementary. An obtuse angle measures more than 90°, and neither does. The author pictured two unequal angles; the counterexample is the case where the

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two are equal.

(b) The segment from $(-1, 0)$ to $(1, 0)$ has midpoint $\left(\frac{-1+1}{2}, 0\right) = (0, 0)$ and length 2. The segment from $(-2, 0)$ to $(2, 0)$ has midpoint $(0, 0)$ as well, and length 4. Same midpoint, lengths $2 \neq 4$.

(c) Take a box with bottom face $ABCD$ and top face $EFGH$, with E above A, F above B, G above C and H above D. The line $\overleftrightarrow{AB}$ runs along the front of the bottom face, and $\overleftrightarrow{CG}$ is the upright edge at the back right. They never meet: $\overleftrightarrow{AB}$ stays in the floor, and the only floor point of $\overleftrightarrow{CG}$ is C, which is not on $\overleftrightarrow{AB}$. They are not parallel either, because parallel lines lie in one plane, and no plane holds both. They are skew lines.

5. Claim: for any three different lines ℓ, m and n, if $\ell \perp m$ and $m \perp n$, then $\ell \parallel n$.

A box has bottom face $ABCD$ and top face $EFGH$, with E above A, F above B, G above C and H above D.

(a) Use lines through edges of the box to give a counterexample, and say why it breaks the claim.

(b) Your counterexample uses lines that do not all lie in one plane. Does it show that the claim is false for three lines that lie in one plane? Explain.

ANSWER

(a) Many answers; for example $\ell = \overleftrightarrow{AB}$, $m = \overleftrightarrow{AE}$, $n = \overleftrightarrow{AD}$: $\ell \perp m$ and $m \perp n$, but ℓ and n meet at A, so they are not parallel. (b) No. A counterexample refutes only the claim as stated, about lines anywhere in space; the narrower claim about three lines in one plane is a different claim and needs its own proof or counterexample.

SOLUTION

(a) Take the three edges that meet at corner A: $\ell = \overleftrightarrow{AB}$, $m = \overleftrightarrow{AE}$ and $n = \overleftrightarrow{AD}$. They are three different lines. Each face of a box is a rectangle, so $\overline{AB} \perp \overline{AE}$ (face $ABFE$) and $\overline{AE} \perp \overline{AD}$ (face $ADHE$). The hypothesis holds. But ℓ and n share the point A, and parallel lines never meet, so $\ell \parallel n$ is false. Hypothesis true, conclusion false: the claim is false. (Another counterexample: $\overleftrightarrow{AB} \perp \overleftrightarrow{BF}$ and $\overleftrightarrow{BF} \perp \overleftrightarrow{FG}$, but $\overleftrightarrow{AB}$ and $\overleftrightarrow{FG}$ are skew, so not parallel.)

(b) No. A counterexample kills the claim as stated, and nothing more. Lines that lie in one plane are a narrower case, and the counterexample in (a) is not in that case, so it says nothing about it either way. Deciding the narrower claim takes its own argument. (Drawn in one plane, ℓ and n both make right angles with m, and the narrower claim is in fact true; the claim's author pictured flat figures and forgot space.)

6. Claim: for every real number x, if $x^2 = 5x$, then $x = 5$.

Owen's proof: "Start from $x^2 = 5x$. Divide both sides by x. This gives $x = 5$."

(a) Which step of Owen's proof needs something extra about x? What does it need?

(b) Use your answer to (a) to find a counterexample to the claim, and check it.

(c) Change the hypothesis of the claim so that the claim becomes true.

ANSWER

(a) Dividing by x needs $x \neq 0$. (b) $x = 0$: $0^2 = 5 \cdot 0$ is true, but $0 \neq 5$. (c) For every nonzero real number x, if $x^2 = 5x$, then $x = 5$.

SOLUTION

(a) Dividing both sides of an equation by x is allowed only when $x \neq 0$. Owen used that without saying so.

(b) The step fails only at $x = 0$, so try it. The hypothesis $0^2 = 5 \cdot 0$ reads $0 = 0$, which is true. The conclusion $0 = 5$ is false. Hypothesis true, conclusion false: $x = 0$ is a counterexample, and the claim is false.

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(c) Add the missing fact to the hypothesis: "for every nonzero real number x, if $x^2 = 5x$, then $x = 5$." Now Owen's division is allowed, and his proof works. (Another true repair keeps every real x and widens the conclusion: if $x^2 = 5x$, then $x = 0$ or $x = 5$, since $x^2 - 5x = x(x - 5)$.)

7. Claim: for all statements p and q, the statement $\neg(p \wedge q)$ is logically equivalent to $\neg p \wedge \neg q$. Find a row of the truth table that is a counterexample, and show the truth value of each statement on that row.

ANSWER

The row with p true and q false: $\neg(p \wedge q)$ is true and $\neg p \wedge \neg q$ is false. (The row with p false and q true works too.)

SOLUTION

The two statements are equivalent only if they agree on every row, so one row where they differ is a counterexample.

The statement $\neg p \wedge \neg q$ is false as soon as one of p, q is true. The statement $\neg(p \wedge q)$ is true as soon as one of p, q is false. So make exactly one of them true: take p true and q false.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	F	F	T	F	T	F

On this row the first statement is true and the second false, so they are **not** equivalent. In words: "not both" allows one of them to hold, and "neither" does not.

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