Geometry Unit 1 worksheet: Deductive reasoning (answer key)

ANSWERS AT A GLANCE

1. (a) $GK = KH$, law of detachment. (b) If an angle measures 30°, then its supplement is obtuse; law of syllogism. (c) Nothing follows. (d) Nothing follows. (e) Nothing follows.
2. $x = 16$; $m\angle AOB = 88°$, $m\angle BOC = 48°$, $m\angle COD = 44°$.
3. (a) (i) Yes. (ii) No: the first statement is false, and so is the conclusion. (b) (i) No: it affirms the consequent. (ii) The conclusion happens to be true, by the definition of a rectangle, but this argument does not show it. (c) (i) Yes. (ii) Yes: true statements and valid steps.
4. (a) See the solution. (b) The line $m\angle ABC = 90°$ becomes the given $m\angle ABC = 130°$, and every later 90 becomes 130; the proof now concludes $m\angle ABD = 65°$.
5. (a) $AN = 15$. (b) $AN = \frac{3}{4}AB$. (c) No. The formula gives 7.5 cm; the gap comes from the drawing and the ruler, not from the proof.
6. See the proof below.
7. (a) $m\angle BIC = 90° + \frac{1}{2}m\angle A$; inductive reasoning. (b) See the solution. (c) $m\angle A = 36°$.

WORKED SOLUTIONS

1. Each part gives statements to accept as true. Write the conclusion that follows and name the law used (law of detachment or law of syllogism). If no conclusion follows, say so and explain why.

(a) If a point is the midpoint of a segment, then it divides the segment into two segments of equal length. K is the midpoint of $\overline{GH}$.

(b) If an angle measures 30°, then it is acute. If an angle is acute, then its supplement is obtuse.

(c) If a triangle is equilateral, then it is isosceles. $\triangle RST$ is isosceles.

(d) If a polygon is a regular hexagon, then each of its angles measures 120°. Polygon W is not a regular hexagon.

(e) If a quadrilateral is a rectangle, then its diagonals are congruent. If a quadrilateral is a kite, then its diagonals are perpendicular.

> **ANSWER**
> (a) $GK = KH$, law of detachment. (b) If an angle measures 30°, then its supplement is obtuse; law of syllogism. (c) Nothing follows. (d) Nothing follows. (e) Nothing follows.
>
> **SOLUTION**
> (a) The second sentence supplies the if-part of the conditional, so the then-part follows: $GK = KH$. This is the **law of detachment**.
>
> (b) The then-part of the first conditional is the if-part of the second, so they join end to end: **if an angle measures 30°, then its supplement is obtuse.** This is the **law of syllogism**.
>
> (c) **Nothing follows.** "Isosceles" is the then-part, not the if-part. Concluding that $\triangle RST$ is equilateral would be affirming the consequent. A triangle with sides 5, 5 and 8 is isosceles but not equilateral.
>
> (d) **Nothing follows.** The conditional says nothing about polygons that are not regular hexagons. W could be a hexagon whose angles all measure 120° but whose sides are not all equal, or a square, whose angles measure 90°.
>
> (e) **Nothing follows.** The then-part of one conditional (congruent diagonals) is not the if-part of the other (being a kite), so the two do not chain, and neither if-part is supplied.

2. Points A, O and D lie on a line, with O between A and D. Rays $\overrightarrow{OB}$ and $\overrightarrow{OC}$ lie on the same side of the line, so that the rays come in the order $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$. $m\angle AOB = (4x + 24)°$, $m\angle BOC = (3x)°$ and $m\angle COD = (3x - 4)°$. Find x and the three angle measures, giving a reason for each step.

> **ANSWER**
> $x = 16$; $m\angle AOB = 88°$, $m\angle BOC = 48°$, $m\angle COD = 44°$.

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SOLUTION

$\angle AOB$ and $\angle BOD$ form a linear pair, since $\overrightarrow{OA}$ and $\overrightarrow{OD}$ are opposite rays. By the Linear Pair Theorem, $m\angle AOB + m\angle BOD = 180°$.

C is in the interior of $\angle BOD$, so by the angle addition postulate, $m\angle BOD = m\angle BOC + m\angle COD$.

Substitution gives $m\angle AOB + m\angle BOC + m\angle COD = 180°$, so

$$(4x + 24) + 3x + (3x - 4) = 180.$$

Combine like terms: $10x + 24 - 4 = 180$, so $10x = 160$ and $x = 16$.

Then $m\angle AOB = 88°$, $m\angle BOC = 48°$ and $m\angle COD = 44°$. Check: 88 + 48 + 44 = 180.

3. For each argument, say (i) whether every step follows from the statements before it, and (ii) whether the conclusion is certain to be true. Explain each answer.

(a) Every angle that measures less than 100° is acute. $m\angle K = 95°$. So $\angle K$ is acute.

(b) Every rectangle has four right angles. Quadrilateral $WXYZ$ has four right angles. So $WXYZ$ is a rectangle.

(c) Every point on the perpendicular bisector of a segment is the same distance from the segment's two endpoints. Point P lies on the perpendicular bisector of $\overline{AB}$. So $PA = PB$.

ANSWER

(a) (i) Yes. (ii) No: the first statement is false, and so is the conclusion. (b) (i) No: it affirms the consequent. (ii) The conclusion happens to be true, by the definition of a rectangle, but this argument does not show it. (c) (i) Yes. (ii) Yes: true statements and valid steps.

SOLUTION

(a) (i) The step is valid: 95 < 100 supplies the if-part, and the law of detachment gives the then-part. (ii) The conclusion is **not** certain, and in fact it is false: an acute angle measures less than 90°, so the first statement is false and a 95° angle is obtuse. Deduction carries truth forward, but it cannot make a false starting statement true.

(b) (i) The step is **not** valid. "Four right angles" is the then-part of the first statement, and the argument claims its if-part, which is affirming the consequent. (ii) The conclusion is true, but for a different reason: a rectangle is defined as a quadrilateral with four right angles, so $WXYZ$ is a rectangle by the definition. A true conclusion does not make an invalid argument valid.

(c) (i) Valid: the second statement supplies the if-part, so the law of detachment gives $PA = PB$. (ii) Certain: the first statement is a theorem the course has proved, and the second is given. True starting statements and valid steps make the conclusion true.

4. **Given:** $\angle ABC$ is a right angle, D is in the interior of $\angle ABC$, and $m\angle ABD = m\angle DBC$. **Prove:** $m\angle ABD = 45°$.

The seven statements of the proof are below, out of order.

- $2\,m\angle ABD = 90°$
- $m\angle ABD + m\angle DBC = m\angle ABC$
- $m\angle ABC = 90°$
- $m\angle ABD = 45°$
- $m\angle ABD + m\angle ABD = 90°$
- $m\angle ABD = m\angle DBC$
- $\angle ABC$ is a right angle

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(a) Put the statements in order and give the reason for each.

(b) Change only the first given, to "$m\angle ABC = 130^\circ$". Which lines of the proof change, and what does the proof now conclude?

ANSWER

(a) See the solution. (b) The line $m\angle ABC = 90^\circ$ becomes the given $m\angle ABC = 130^\circ$, and every later 90 becomes 130; the proof now concludes $m\angle ABD = 65^\circ$.

SOLUTION

(a)

Statements	Reasons
1. $\angle ABC$ is a right angle	given
2. $m\angle ABC = 90^\circ$	definition of right angle: line 1
3. $m\angle ABD + m\angle DBC = m\angle ABC$	angle addition postulate: D is in the interior of $\angle ABC$
4. $m\angle ABD = m\angle DBC$	given
5. $m\angle ABD + m\angle ABD = 90^\circ$	substitution: lines 2 and 4 into line 3
6. $2\,m\angle ABD = 90^\circ$	simplify: line 5
7. $m\angle ABD = 45^\circ$	divide both sides by 2 (division property of equality): line 6

Lines 1 to 4 can come in other orders, as long as line 2 follows line 1; lines 5, 6 and 7 must come last, in that order, because each uses the one before.

(b) Lines 1 and 2 are replaced by the single given $m\angle ABC = 130^\circ$. Lines 3 and 4 do not change. Line 5 becomes $m\angle ABD + m\angle ABD = 130^\circ$, line 6 becomes $2\,m\angle ABD = 130^\circ$, and the conclusion becomes $m\angle ABD = 65^\circ$. The chain of reasons is the same; only the number it carries changes.

5. M is the midpoint of $\overline{AB}$, and N is the midpoint of $\overline{MB}$.

(a) If $AB = 20$, find AN, giving a reason for each step.

(b) Find a formula for AN in terms of AB that holds for every segment $\overline{AB}$, and prove it.

(c) Maya draws the figure carefully and measures $AB = 10$ cm and $AN = 7.3$ cm. Does her measurement cast doubt on your answer to (b)? Explain.

ANSWER

(a) $AN = 15$. (b) $AN = \dfrac{3}{4}AB$. (c) No. The formula gives 7.5 cm; the gap comes from the drawing and the ruler, not from the proof.

SOLUTION

(a) M is the midpoint of $\overline{AB}$, so $AM = MB$ (definition of midpoint). $AM + MB = AB$ (segment addition postulate), so $2AM = 20$ and $AM = MB = 10$. In the same way, N is the midpoint of $\overline{MB}$, so $MN = NB$ and $MN + NB = MB = 10$, which gives $MN = 5$. N is between M and B, so M is between A and N, and $AN = AM + MN$ (segment addition postulate) = 10 + 5, so $AN = 15$.

(b) Let $AB = s$. The same steps with s in place of 20 give $AM = MB = \dfrac{s}{2}$ (definition of midpoint and segment addition postulate), then $MN = NB = \dfrac{1}{2} \cdot \dfrac{s}{2} = \dfrac{s}{4}$ (the same two reasons for $\overline{MB}$), and

$$AN = AM + MN = \frac{s}{2} + \frac{s}{4} = \frac{3s}{4}$$

by the segment addition postulate and substitution. So $AN = \dfrac{3}{4}AB$. The argument never used a length read off a figure, so it holds for every segment.

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(c) No. By (b), $AN = \frac{3}{4}(10) = 7.5$ cm. The 0.2 cm gap comes from where her points and lines landed on the paper and from reading the ruler. A measurement is evidence about one drawing; the deduction covers every segment, and a drawing cannot overturn it.

6. Two lines cross at O and form four angles, $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$, in that order around O. So each angle forms a linear pair with the angle before it and the angle after it.

Given: $\angle 1 \cong \angle 2$.

Prove: all four angles are right angles.

Write a two-column proof. Do not use the Vertical Angles Theorem.

ANSWER

See the two-column proof in the solution.

SOLUTION

Statements	Reasons
1. $\angle 1 \cong \angle 2$	given
2. $m\angle 1 = m\angle 2$	definition of congruent angles: line 1
3. $m\angle 1 + m\angle 2 = 180°$	Linear Pair Theorem: $\angle 1$ and $\angle 2$ form a linear pair
4. $m\angle 1 + m\angle 1 = 180°$	substitution: line 2 into line 3
5. $m\angle 1 = 90°$	simplify and divide both sides by 2: line 4
6. $m\angle 2 = 90°$	substitution: line 5 into line 2
7. $m\angle 2 + m\angle 3 = 180°$	Linear Pair Theorem: $\angle 2$ and $\angle 3$ form a linear pair
8. $m\angle 3 = 90°$	substitution: line 6 into line 7, then subtract 90°
9. $m\angle 4 + m\angle 1 = 180°$	Linear Pair Theorem: $\angle 4$ and $\angle 1$ form a linear pair
10. $m\angle 4 = 90°$	substitution: line 5 into line 9, then subtract 90°
11. $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$ are right angles	definition of right angle: lines 5, 6, 8 and 10

No angle was measured, so the result holds for every pair of crossing lines in which two neighboring angles are congruent.

7. Take the angle sum theorem as already proved: the three angles of any triangle add to 180°. A ray **bisects** an angle when it splits the angle into two angles of equal measure.

In $\triangle ABC$, the ray that bisects $\angle ABC$ and the ray that bisects $\angle ACB$ meet at a point I inside the triangle.

(a) Three such triangles are drawn and measured.

Triangle	$m\angle A$	$m\angle BIC$
1	40°	110°
2	70°	125°
3	100°	140°

State a conjecture that gives $m\angle BIC$ in terms of $m\angle A$. What kind of reasoning produced it?

(b) Prove your conjecture for every triangle, naming each accepted fact you use.

(c) In one such triangle, $m\angle BIC$ is three times $m\angle A$. Find $m\angle A$.

ANSWER

(a) $m\angle BIC = 90° + \frac{1}{2}m\angle A$; inductive reasoning. (b) See the solution. (c) $m\angle A = 36°$.

SOLUTION

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(a) Each time $m\angle A$ goes up by 30°, $m\angle BIC$ goes up by 15°, half as much, and $110 = 90 + \frac{40}{2}$, $125 = 90 + \frac{70}{2}$, $140 = 90 + \frac{100}{2}$. Conjecture: **in every triangle,** $m\angle BIC = 90^\circ + \frac{1}{2} m\angle A$**.** Three measured examples give a pattern, so this is **inductive** reasoning, and the result is only a conjecture so far.
(b) Take any $\triangle ABC$ and let $m\angle A = a^\circ$.

- $m\angle ABC + m\angle ACB = 180^\circ - a^\circ$, by the angle sum theorem in $\triangle ABC$.
- $\overrightarrow{BI}$ bisects $\angle ABC$, so $m\angle IBC = \frac{1}{2} m\angle ABC$ (definition of bisects). In the same way, $m\angle ICB = \frac{1}{2} m\angle ACB$.
- Adding and substituting: $m\angle IBC + m\angle ICB = \frac{1}{2}(m\angle ABC + m\angle ACB) = \frac{1}{2}(180^\circ - a^\circ) = 90^\circ - \frac{a}{2}$ degrees.
- By the angle sum theorem in $\triangle BIC$, $m\angle BIC = 180^\circ - (m\angle IBC + m\angle ICB) = 180^\circ - \left(90^\circ - \frac{a}{2}\right) = 90^\circ + \frac{a}{2}$ degrees.

No measure was read from a figure, so this covers every triangle, and the conjecture is now a theorem.
(c) By (b), $90 + \frac{a}{2} = 3a$, so $90 = \frac{5a}{2}$ and $a = 36$. So $m\angle A = 36^\circ$. Check: $m\angle BIC = 90^\circ + 18^\circ = 108^\circ = 3 \cdot 36^\circ$.

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