Geometry Unit 1 worksheet: The midpoint formula (answer key)

ANSWERS AT A GLANCE

1. (a) $B = (10, 11)$ (b) $B = (5, -8)$ (c) $B = (-7, 7)$ (d) $B = (4.5, -3)$ (e) $B = \left(-\frac{7}{2}, 5\right)$ (f) $A = (-8, 3)$
2. (a) $M = (5, 6)$ (b) $AM = 10$
3. (a) $E = (4, 9)$ (b) 200 m
4. (a) The midpoint of A and (3.5, −3.5) is (2.75, −4.75), not M. (b) The midpoint of $\overline{AM}$, a quarter of the way from A to B. (c) $B = (8, 4)$
5. (a) $k = 4$; $B = (11, 9)$ and $M = (6, 8)$. (b) No: the x-coordinates need $k = 6$ but the y-coordinates need $k = 3$.
6. (a) No: it is perpendicular, but it misses the midpoint $\left(2, \frac{5}{2}\right)$. (b) Yes. (c) No: it passes through the midpoint $(0, 2)$, but it is not perpendicular.
7. (a) Center $4 + i$, radius $3\sqrt{2}$. (b) $1 - 2i$
8. (a) $P' = (4, 1)$, $P'' = (4, -3)$. (b) P to P'' is 6 right and 8 down, exactly twice the move from C to D, which is 3 right and 4 down. (c) $P'' = 2D - P' = 2D - (2C - P) = P + 2(D - C)$.
9. (−1, 5), (3, −3) and (7, 7)

WORKED SOLUTIONS

1. In each part M is the midpoint of $\overline{AB}$. Find the missing endpoint.

(a) $A = (2, 5)$, $M = (6, 8)$ (b) $A = (-3, 4)$, $M = (1, -2)$ (c) $A = (7, -1)$, $M = (0, 3)$

(d) $A = (-2.5, 6)$, $M = (1, 1.5)$ (e) $A = \left(\frac{3}{2}, -4\right)$, $M = \left(-1, \frac{1}{2}\right)$ (f) $B = (4, -9)$, $M = (-2, -3)$

ANSWER

(a) $B = (10, 11)$ (b) $B = (5, -8)$ (c) $B = (-7, 7)$ (d) $B = (4.5, -3)$ (e) $B = \left(-\frac{7}{2}, 5\right)$ (f) $A = (-8, 3)$

SOLUTION

Each missing coordinate is twice the midpoint's coordinate minus the known end's: $x_2 = 2x_M - x_1$ and $y_2 = 2y_M - y_1$.

(a) (2 · 6 − 2, 2 · 8 − 5) = (10, 11).

(b) (2 · 1 − (−3), 2(−2) − 4) = (5, −8).

(c) (2 · 0 − 7, 2 · 3 − (−1)) = (−7, 7).

(d) (2 · 1 − (−2.5), 2 · 1.5 − 6) = (4.5, −3).

(e) $\left(2(-1) - \frac{3}{2},\ 2 \cdot \frac{1}{2} - (-4)\right) = \left(-\frac{7}{2}, 5\right)$.

(f) Here the known end is B, and the same rule gives the other end: (2(−2) − 4, 2(−3) − (−9)) = (−8, 3).

Check any part by averaging: in (f), $\left(\frac{-8 + 4}{2}, \frac{3 + (-9)}{2}\right) = (-2, -3)$, which is M.

2. A **median** of a triangle runs from a corner to the midpoint of the opposite side. Triangle ABC has $A = (-1, -2)$, $B = (8, 3)$ and $C = (2, 9)$.

(a) Find the midpoint M of $\overline{BC}$.

(b) Find the length of the median $\overline{AM}$.

ANSWER

(a) $M = (5, 6)$ (b) $AM = 10$

SOLUTION

(a) $M = \left(\frac{8 + 2}{2}, \frac{3 + 9}{2}\right) = (5, 6)$.

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(b) By the distance formula,

$$AM = \sqrt{(5-(-1))^2 + (6-(-2))^2} = \sqrt{36+64} = \sqrt{100} = 10$$

3. On a park map, one grid unit stands for 20 m. A straight boardwalk across a marsh starts at the visitor center $V = (-4, 3)$. A bird hide stands at the exact middle of the boardwalk, at $H = (0, 6)$.

(a) Find the coordinates of the far end E of the boardwalk.

(b) How long is the boardwalk, in meters?

ANSWER

(a) $E = (4, 9)$ (b) 200 m

SOLUTION

(a) From V to H the x-coordinate changes by 4 and the y-coordinate by 3. H is halfway, so E is the same change again, starting at H:

$$E = (0+4,\ 6+3) = (4, 9)$$

Check: the midpoint of V and E is $\left(\frac{-4+4}{2}, \frac{3+9}{2}\right) = (0, 6)$, which is H.

(b) By the distance formula, $VH = \sqrt{4^2 + 3^2} = 5$ grid units. The boardwalk is twice as long as VH, so it is 10 grid units, which is $10 \cdot 20 = 200$ m.

4. Diego is told that $M = (5, -1)$ is the midpoint of $\overline{AB}$ and that $A = (2, -6)$. To find B, he averages A and M and gets (3.5, −3.5).

(a) Show that Diego's point cannot be B.

(b) Diego did find the midpoint of some segment. Which segment?

(c) Find B.

ANSWER

(a) The midpoint of A and (3.5, −3.5) is (2.75, −4.75), not M. (b) The midpoint of $\overline{AM}$, a quarter of the way from A to B. (c) $B = (8, 4)$

SOLUTION

(a) If (3.5, −3.5) were B, the midpoint of $\overline{AB}$ would be $\left(\frac{2+3.5}{2}, \frac{-6+(-3.5)}{2}\right) = (2.75, -4.75)$. That is not $M = (5, -1)$, so Diego's point is not B.

(b) Averaging two points gives the midpoint of the segment joining them. Diego averaged A and M, so he found the midpoint of $\overline{AM}$, which is halfway from A to M, a quarter of the way from A to B.

(c) M is halfway, so B is found by running the formula backwards, not by averaging:

$$B = (2 \cdot 5 - 2,\ 2(-1) - (-6)) = (8, 4)$$

Check: $\left(\frac{2+8}{2}, \frac{-6+4}{2}\right) = (5, -1)$, which is M.

5. (a) The point $M = (k+2, 2k)$ is the midpoint of the segment from $A = (1, 7)$ to $B = (3k-1, k+5)$. Find k. Then use the other coordinate to check that the same k works, and give B and M.

(b) Can the point (k, k) be the midpoint of the segment from (2, 6) to $(k+4, 0)$ for some number k? Explain.

ANSWER

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(a) $k = 4$; $B = (11, 9)$ and $M = (6, 8)$. (b) No: the x-coordinates need $k = 6$ but the y-coordinates need $k = 3$.

SOLUTION

(a) The x-coordinates give

$$\frac{1 + (3k - 1)}{2} = k + 2, \qquad 3k = 2k + 4, \qquad k = 4$$

Check with the y-coordinates: $\dfrac{7 + (k + 5)}{2} = \dfrac{12 + 4}{2} = 8$, and $2k = 8$. Both coordinates agree, so $k = 4$. Then $B = (3 \cdot 4 - 1, 4 + 5) = (11, 9)$ and $M = (6, 8)$.

(b) The x-coordinates need $\dfrac{2 + (k + 4)}{2} = k$, so $k + 6 = 2k$ and $k = 6$. The y-coordinates need $\dfrac{6 + 0}{2} = k$, so $k = 3$. One number cannot be both 6 and 3, so no value of k works: (k, k) is never the midpoint.

6. Decide whether the line is the perpendicular bisector of the segment. If it is not, say which condition fails: passing through the midpoint, or being perpendicular to the segment.

(a) The line $y = 2x - 3$ and the segment from $(-1, 4)$ to $(5, 1)$.

(b) The line $y = -x + 5$ and the segment from $(-2, 1)$ to $(4, 7)$.

(c) The line $y = \dfrac{1}{3}x + 2$ and the segment from $(-3, 0)$ to $(3, 4)$.

ANSWER

(a) No: it is perpendicular, but it misses the midpoint $\left(2, \dfrac{5}{2}\right)$. (b) Yes. (c) No: it passes through the midpoint $(0, 2)$, but it is not perpendicular.

SOLUTION

(a) Midpoint $\left(\dfrac{-1 + 5}{2}, \dfrac{4 + 1}{2}\right) = \left(2, \dfrac{5}{2}\right)$. At $x = 2$ the line gives $y = 2(2) - 3 = 1 \neq \dfrac{5}{2}$, so the line misses the midpoint. The slope of the segment is $\dfrac{1 - 4}{5 - (-1)} = -\dfrac{1}{2}$, and $2 \cdot \left(-\dfrac{1}{2}\right) = -1$, so the line is perpendicular. It fails only the midpoint condition.

(b) Midpoint $\left(\dfrac{-2 + 4}{2}, \dfrac{1 + 7}{2}\right) = (1, 4)$, and $-1 + 5 = 4$, so the line passes through it. The slope of the segment is $\dfrac{7 - 1}{4 - (-2)} = 1$, and $1 \cdot (-1) = -1$, so the line is perpendicular. Both conditions hold: it is the perpendicular bisector.

(c) Midpoint $\left(\dfrac{-3 + 3}{2}, \dfrac{0 + 4}{2}\right) = (0, 2)$, and $\dfrac{1}{3}(0) + 2 = 2$, so the line passes through it. The slope of the segment is $\dfrac{4 - 0}{3 - (-3)} = \dfrac{2}{3}$, and $\dfrac{1}{3} \cdot \dfrac{2}{3} = \dfrac{2}{9} \neq -1$, so the line is not perpendicular. It fails only the perpendicular condition.

7. In the complex plane, the numbers $1 + 4i$ and $7 - 2i$ are the ends of a diameter of a circle.

(a) Find the center of the circle, as a complex number, and the radius.

(b) The number $7 + 4i$ is also on the circle. Find the other end of the diameter that starts at $7 + 4i$.

ANSWER

(a) Center $4 + i$, radius $3\sqrt{2}$. (b) $1 - 2i$

SOLUTION

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(a) The center is the midpoint of the diameter: $\dfrac{(1+4i)+(7-2i)}{2} = \dfrac{8+2i}{2} = 4+i$. The radius is the distance from (4, 1) to (1, 4): $\sqrt{(1-4)^2+(4-1)^2} = \sqrt{18} = 3\sqrt{2}$.
(b) The distance from (4, 1) to (7, 4) is $\sqrt{3^2+3^2} = 3\sqrt{2}$, so $7+4i$ is on the circle. The center is the midpoint of the new diameter, so the other end is $2(4+i)-(7+4i) = 8+2i-7-4i = 1-2i$.
Check: $\dfrac{(7+4i)+(1-2i)}{2} = \dfrac{8+2i}{2} = 4+i$.

8. Reflecting a point P **through** a point C gives the point P' for which C is the midpoint of $\overline{PP'}$.

(a) Reflect $P = (-2, 5)$ through $C = (1, 3)$ to get P'. Then reflect P' through $D = (4, -1)$ to get P''.

(b) Compare the move from P to P'' with the move from C to D.

(c) Explain why the comparison in (b) holds for every starting point P and every two points C and D.

ANSWER
(a) $P' = (4, 1)$, $P'' = (4, -3)$. (b) P to P'' is 6 right and 8 down, exactly twice the move from C to D, which is 3 right and 4 down. (c) $P'' = 2D - P' = 2D - (2C - P) = P + 2(D - C)$.

SOLUTION
(a) C is the midpoint of $\overline{PP'}$, so $P' = (2\cdot 1 - (-2),\ 2\cdot 3 - 5) = (4, 1)$. D is the midpoint of $\overline{P'P''}$, so $P'' = (2\cdot 4 - 4,\ 2(-1) - 1) = (4, -3)$.

(b) From $P = (-2, 5)$ to $P'' = (4, -3)$ is 4 – (−2) = 6 right and −3 − 5 = −8, that is 8 down. From C to D is 3 right and 4 down. The first move is exactly twice the second.

(c) Work one coordinate at a time; the y-coordinates go the same way. Running the midpoint formula backwards, $x_{P'} = 2x_C - x_P$ and $x_{P''} = 2x_D - x_{P'}$. Put the first into the second:

$$x_{P''} = 2x_D - (2x_C - x_P) = x_P + 2(x_D - x_C)$$

So the move across from P to P'' is $2(x_D - x_C)$, twice the move across from C to D, whatever P, C and D are.

9. Three corners of a parallelogram are $A = (1, 1)$, $B = (5, 2)$ and $C = (3, 6)$, but you are not told the order of the corners around the shape. The fourth corner D can be in three different places. Find all three.

(Hint: in each parallelogram, one of A, B, C is the corner opposite D, and the diagonals bisect each other.)

ANSWER
(−1, 5), (3, −3) and (7, 7)

SOLUTION
The diagonals of a parallelogram bisect each other, so the two corners not opposite D form one diagonal, and its midpoint is also the midpoint of the corner opposite D and D itself.

D opposite B. The diagonal $\overline{AC}$ has midpoint $\left(\dfrac{1+3}{2}, \dfrac{1+6}{2}\right) = \left(2, \dfrac{7}{2}\right)$. Running it backwards from B: $D = \left(2\cdot 2 - 5,\ 2\cdot \dfrac{7}{2} - 2\right) = (-1, 5)$.

D opposite C. The diagonal $\overline{AB}$ has midpoint $\left(3, \dfrac{3}{2}\right)$. From C: $D = \left(2\cdot 3 - 3,\ 2\cdot \dfrac{3}{2} - 6\right) = (3, -3)$.

D opposite A. The diagonal $\overline{BC}$ has midpoint (4, 4). From A: $D = (2\cdot 4 - 1,\ 2\cdot 4 - 1) = (7, 7)$.

Check one: in $ABCD$ with $D = (-1, 5)$, from A to B is 4 right and 1 up, and from D to C is also 4 right and 1 up.

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