Geometry Unit 1 worksheet: Inductive reasoning (answer key)

ANSWERS AT A GLANCE

1. (a) 2, 4 and 6 regions. (b) No: both rules give 2 and 4. Three lines give $6 = 2 \cdot 3$ regions, not $2^3 = 8$, so Maya's conjecture is false and Diego's survives. (c) See the solution: n lines make $2n$ regions. (d) 24 regions.
2. (a) 1, 3, 6, 10. (b) The differences are 2, 3, 4, so the next is 5: predicted $10 + 5 = 15$. (c) $\frac{n(n+1)}{2}$ rectangles; for $n = 5$ it gives 15, which agrees. (d) 465 rectangles.
3. (a) Yes, a counterexample. (b) No: it meets the hypothesis and is a parallelogram. (c) No: it does not meet the hypothesis, since no pair of its opposite sides is parallel.
4. (a) One answer: $A(0,0)$, $B(5,0)$, $C(3,4)$, $D(0,4)$, with right angles at A and D but not at B. (b) $\angle Q$ and $\angle S$ are right but $\angle P$ is not, so Owen's repair is false. (c) A convex quadrilateral with three right angles is a rectangle; its fourth angle is $360° - 270° = 90°$.
5. (a) One answer: each is a parallelogram (both pairs of opposite sides parallel). Other true answers, such as a line of symmetry or diagonals that bisect each other, also earn credit. (b) $O = (2,2)$, and the areas are 6, 3, $\frac{3}{2}$ and 3. They are not equal, so $ABCD$ is a counterexample and the claim is false. (c) One answer: the diagonals of every parallelogram cut it into four triangles of equal area.
6. (a) (i) 24 and 6; (ii) 30 and $\frac{15}{2}$; (iii) 12 and 3. (b) The midpoint triangle of every triangle has one quarter of its area. The three test triangles differ: (i) is acute, (ii) right and (iii) obtuse. (c) The areas are $\frac{bd}{2}$ and $\frac{bd}{8}$. (d) Every triangle can be moved into that position.
7. (a) 6, 24 and 12. (b) Its area is $\sqrt{3}$, which is not a whole number. (c) Its area is $\frac{3\sqrt{7}}{4}$, not a whole number, so the repair is false. It ruled out only the kind of triangle that broke the claim, not the reason the claim fails.

WORKED SOLUTIONS

1. Different lines, all passing through one point P, are drawn in a plane. They cut the plane into regions.

(a) Draw 1, 2 and 3 lines through P, and count the regions each time.

(b) Maya looked only at 1 and 2 lines and conjectured that n lines make 2^n regions. Diego looked at the same two cases and conjectured $2n$ regions. Can the cases of 1 and 2 lines decide between them? What does your count for 3 lines decide?

(c) Give a reason that covers every n.

(d) How many regions do 12 lines through P make?

ANSWER

(a) 2, 4 and 6 regions. (b) No: both rules give 2 and 4. Three lines give $6 = 2 \cdot 3$ regions, not $2^3 = 8$, so Maya's conjecture is false and Diego's survives. (c) See the solution: n lines make $2n$ regions. (d) 24 regions.

SOLUTION

(a) One line makes 2 regions (two half-planes), two lines make 4, and three lines make 6.

(b) Maya's rule gives $2^1 = 2$ and $2^2 = 4$; Diego's gives $2 \cdot 1 = 2$ and $2 \cdot 2 = 4$. Both fit the two cases, so those cases cannot decide between them: any finite list of cases fits more than one rule. Three lines give 6 regions. That is $2 \cdot 3 = 6$, but $2^3 = 8$, so the case of 3 lines is a counterexample to Maya's conjecture. Diego's conjecture survives, which is not yet a proof.

(c) Each line through P is made of two rays that start at P. The lines are different, so n lines give $2n$ different rays from P. Going once around P, the rays come one after another, and between each ray and the next there is exactly one region. So there are exactly as many regions as rays: $2n$. The argument never used a particular value of n, so it covers every n at once, and Diego's conjecture is now proved.

(d) $2 \cdot 12 = 24$ regions.

2. A **strip** is a row of n unit squares side by side. A rectangle in the strip is made of one or more neighboring whole squares of the strip; a single square counts as a rectangle.

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(a) Count the rectangles in strips of 1, 2, 3 and 4 squares.

(b) Use the differences between your counts to predict the count for a strip of 5 squares.

(c) Give a reason that covers every n, and use it to write the count as a formula in n. Does it agree with your prediction?

(d) How many rectangles are in a strip of 30 squares?

ANSWER

(a) 1, 3, 6, 10. (b) The differences are 2, 3, 4, so the next is 5: predicted 10 + 5 = 15. (c) $\dfrac{n(n+1)}{2}$ rectangles; for $n = 5$ it gives 15, which agrees. (d) 465 rectangles.

SOLUTION

(a) A strip of 3 squares has 3 rectangles of length 1, 2 of length 2 and 1 of length 3, so 6. In the same way the counts for 1, 2, 3, 4 squares are 1, 3, 6, 10.

(b) The differences are 2, 3, 4. If they keep growing by 1, the next difference is 5, and the prediction for 5 squares is 10 + 5 = 15. This is a conjecture about one case.

(c) A strip of n squares has $n + 1$ vertical lines, counting both ends. A rectangle in the strip is fixed by its left edge and its right edge, which are two different lines among these, and any two different lines are the edges of exactly one rectangle. Pairing each of the $n + 1$ lines with each of the other n counts every pair twice, once from each end, so there are

$$\frac{(n+1)n}{2}$$

rectangles. For $n = 5$ this is $\dfrac{6 \cdot 5}{2} = 15$, which confirms the prediction. The argument holds for every n, not only the ones counted.

(d) $\dfrac{31 \cdot 30}{2} = 465$ rectangles.

3. Claim: if one pair of opposite sides of a quadrilateral is parallel and the other pair of opposite sides is equal in length, then the quadrilateral is a parallelogram.

Decide whether each quadrilateral is a counterexample to the claim. Show every fact you use exactly, from the coordinates.

(a) $A(0,0)$, $B(8,0)$, $C(6,3)$, $D(2,3)$

(b) $E(0,0)$, $F(6,0)$, $G(8,3)$, $H(2,3)$

(c) $J(0,0)$, $K(6,0)$, $L(9,4)$, $M(0,5)$

ANSWER

(a) Yes, a counterexample. (b) No: it meets the hypothesis and is a parallelogram. (c) No: it does not meet the hypothesis, since no pair of its opposite sides is parallel.

SOLUTION

A counterexample must meet the hypothesis and fail the conclusion. A parallelogram has both pairs of opposite sides parallel.

(a) $\overline{AB}$ and $\overline{DC}$ both have slope 0, so they are parallel. $AD = \sqrt{2^2 + 3^2} = \sqrt{13}$ and $BC = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$, so the other pair is equal. The hypothesis is met. But $\overline{AD}$ has slope $\dfrac{3}{2}$ and $\overline{BC}$ has slope $-\dfrac{3}{2}$, so they are not parallel and $ABCD$ is not a parallelogram. It is a **counterexample** (an isosceles trapezoid).

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(b) $\overline{EF}$ and $\overline{HG}$ both have slope 0. $FG = \sqrt{2^2 + 3^2} = \sqrt{13}$ and $EH = \sqrt{2^2 + 3^2} = \sqrt{13}$. The hypothesis is met. Here $\overline{FG}$ and $\overline{EH}$ both have slope $\dfrac{3}{2}$, so both pairs of opposite sides are parallel and $EFGH$ is a parallelogram. It meets the conclusion too: **not a counterexample**, only one more agreeing case.

(c) The opposite pairs are $\overline{JK}$, $\overline{LM}$ and $\overline{KL}$, $\overline{MJ}$. $\overline{JK}$ has slope 0 and $\overline{LM}$ has slope $\dfrac{5-4}{0-9} = -\dfrac{1}{9}$; $\overline{KL}$ has slope $\dfrac{4}{3}$ and $\overline{MJ}$ is vertical. No pair of opposite sides is parallel, so the hypothesis is not met. $JKLM$ is not a parallelogram, and $KL = MJ = 5$, but a figure that misses a hypothesis proves nothing: **not a counterexample**.

4. Claim: a quadrilateral with two right angles is a rectangle. (A rectangle is a quadrilateral with four right angles.)

(a) Give a counterexample figure by coordinates, and show exactly that it meets the hypothesis and is not a rectangle.

(b) Owen repairs the claim: a quadrilateral with right angles at two **opposite** corners is a rectangle. Test his repair on the convex quadrilateral $P(0,0)$, $Q(1,-2)$, $R(5,0)$, $S(1,2)$.

(c) Repair the claim a different way so that it is true, and prove your repair. You may use that the four angles of a convex quadrilateral add to 360°.

ANSWER

(a) One answer: $A(0,0)$, $B(5,0)$, $C(3,4)$, $D(0,4)$, with right angles at A and D but not at B. (b) $\angle Q$ and $\angle S$ are right but $\angle P$ is not, so Owen's repair is false. (c) A convex quadrilateral with three right angles is a rectangle; its fourth angle is 360° − 270° = 90°.

SOLUTION

(a) Take $A(0,0)$, $B(5,0)$, $C(3,4)$, $D(0,4)$. $\overline{AB}$ and $\overline{DC}$ are horizontal and $\overline{AD}$ is vertical, so $\angle A$ and $\angle D$ are right angles: the hypothesis is met. $\overline{BA}$ is horizontal but $\overline{BC}$ has slope $\dfrac{4-0}{3-5} = -2$, and a horizontal line is perpendicular only to a vertical one, so $\angle B$ is not right. $ABCD$ is not a rectangle.

(b) At Q: $\overline{QP}$ has slope $\dfrac{0-(-2)}{0-1} = -2$ and $\overline{QR}$ has slope $\dfrac{0-(-2)}{5-1} = \dfrac{1}{2}$. The product is −1, so $\angle Q$ is right. At S: $\overline{SP}$ has slope $\dfrac{0-2}{0-1} = 2$ and $\overline{SR}$ has slope $\dfrac{0-2}{5-1} = -\dfrac{1}{2}$, product −1, so $\angle S$ is right. The right angles are at opposite corners, so the hypothesis of the repair is met. At P the slopes are −2 and 2, product −4, so $\angle P$ is not right and $PQRS$ is not a rectangle. Owen's repair is **false**: placing the two right angles differently does not save the claim.

(c) Repair: **a convex quadrilateral with three right angles is a rectangle.**

Proof. The four angles add to 360°. Three of them are 90°, so the fourth is 360° − 3 · 90° = 90°. All four angles are right, so the quadrilateral is a rectangle.

The proof uses the added third right angle: with only two, the other two angles share 180° in any way, as in (a) and (b).

5. Claim: the two diagonals of every convex quadrilateral cut it into four triangles of equal area.

Maya checked a square, a rectangle and a rhombus, and in each one the claim held.

(a) Name a property all three of her figures have that the claim did not ask for.

(b) The quadrilateral $A(0,0)$, $B(6,0)$, $C(3,3)$, $D(0,3)$ is convex. Find the point O where its diagonals meet, and then the areas of triangles OAB, OBC, OCD and ODA. What does this show?

(c) Repair the claim by adding a hypothesis taken from your answer to (a).

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ANSWER

(a) One answer: each is a parallelogram (both pairs of opposite sides parallel). Other true answers, such as a line of symmetry or diagonals that bisect each other, also earn credit. (b) $O = (2, 2)$, and the areas are 6, 3, $\frac{3}{2}$ and 3. They are not equal, so $ABCD$ is a counterexample and the claim is false. (c) One answer: the diagonals of every parallelogram cut it into four triangles of equal area.

SOLUTION

(a) A square, a rectangle and a rhombus are all parallelograms: both pairs of opposite sides are parallel. The claim asked only for a convex quadrilateral.

(b) $\overline{AC}$ lies on $y = x$. $\overline{BD}$ has slope $\frac{3-0}{0-6} = -\frac{1}{2}$ and passes through $D(0, 3)$, so it lies on $y = 3 - \frac{x}{2}$. They meet where $x = 3 - \frac{x}{2}$, so $x = 2$ and $O = (2, 2)$.

$\overline{AB}$ lies on the x-axis, $\overline{DC}$ on $y = 3$ and $\overline{DA}$ on the y-axis, so three of the areas come from a base and a height:

$$\begin{aligned} \text{area } OAB &= \tfrac{1}{2}(6)(2) = 6 \\ \text{area } OCD &= \tfrac{1}{2}(3)(1) = \tfrac{3}{2} \\ \text{area } ODA &= \tfrac{1}{2}(3)(2) = 3 \end{aligned}$$

$ABCD$ is a trapezoid with parallel sides 6 and 3 and height 3, so its area is $\frac{6+3}{2} \cdot 3 = \frac{27}{2}$, and area $OBC = \frac{27}{2} - 6 - \frac{3}{2} - 3 = 3$.

$ABCD$ is a convex quadrilateral, so it meets the hypothesis, and its four triangles are not equal in area. It is a **counterexample**; the claim is false. Its opposite sides $\overline{BC}$ and $\overline{AD}$ are not parallel, and its diagonals do not bisect each other: the midpoint of $\overline{AC}$ is $\left(\frac{3}{2}, \frac{3}{2}\right)$ and that of $\overline{BD}$ is $\left(3, \frac{3}{2}\right)$. Those are features Maya's figures had and this one lacks.

(c) Repair: **the diagonals of every parallelogram cut it into four triangles of equal area.** This is a new conjecture, with its own proof still to come. (It is true.)

6. The **midpoint triangle** of a triangle is the triangle whose corners are the midpoints of its three sides.

(a) For each triangle, find its area and the area of its midpoint triangle.

(i) $A(0, 0)$, $B(8, 0)$, $C(2, 6)$

(ii) $A(0, 0)$, $B(6, 0)$, $C(0, 10)$

(iii) $A(0, 0)$, $B(4, 0)$, $C(10, 6)$

(b) State a conjecture about every triangle. Why are these three triangles a better test than three equilateral triangles would be?

(c) Prove your conjecture for the triangle $A(0, 0)$, $B(b, 0)$, $C(c, d)$ with $b > 0$ and $d > 0$, keeping b, c and d as letters.

(d) Explain why (c) covers every triangle, and not only some.

ANSWER

(a) (i) 24 and 6; (ii) 30 and $\frac{15}{2}$; (iii) 12 and 3. (b) The midpoint triangle of every triangle has one quarter of its area. The three test triangles differ: (i) is acute, (ii) right and (iii) obtuse. (c) The areas are $\frac{bd}{2}$ and $\frac{bd}{8}$. (d) Every triangle can be moved into that position.

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SOLUTION

(a) Each triangle has base $\overline{AB}$ on the x-axis, so its area is $\frac{1}{2} \cdot AB \cdot (\text{height of } C)$. In each midpoint triangle, the midpoints of $\overline{AC}$ and $\overline{BC}$ have the same y-coordinate, so that side is horizontal.

(i) Area $\frac{1}{2}(8)(6) = 24$. Midpoints (4, 0), (5, 3), (1, 3): base 5 − 1 = 4 at height 3, so area $\frac{1}{2}(4)(3) = 6$.

(ii) Area $\frac{1}{2}(6)(10) = 30$. Midpoints (3, 0), (3, 5), (0, 5): base 3 at height 5, so area $\frac{1}{2}(3)(5) = \frac{15}{2}$.

(iii) Area $\frac{1}{2}(4)(6) = 12$. Midpoints (2, 0), (7, 3), (5, 3): base 2 at height 3, so area $\frac{1}{2}(2)(3) = 3$.

(b) Each time the midpoint triangle has $\frac{1}{4}$ of the area. Conjecture: **the midpoint triangle of every triangle has one quarter of the triangle's area.** The cases were chosen to differ: (i) has all angles acute, (ii) a right angle at A, (iii) an obtuse angle at B. Equilateral triangles all have the same shape, so a pattern seen only in them might be a fact about equilateral triangles alone.

(c) The midpoints are $M\left(\frac{b}{2}, 0\right)$ of $\overline{AB}$, $N\left(\frac{b+c}{2}, \frac{d}{2}\right)$ of $\overline{BC}$ and $K\left(\frac{c}{2}, \frac{d}{2}\right)$ of $\overline{CA}$. $\overline{KN}$ is horizontal with length $\frac{b+c}{2} - \frac{c}{2} = \frac{b}{2}$, and M is $\frac{d}{2}$ below it. So

$$\text{area } MNK = \frac{1}{2} \cdot \frac{b}{2} \cdot \frac{d}{2} = \frac{bd}{8}, \qquad \text{area } ABC = \frac{1}{2} \cdot b \cdot d = \frac{bd}{2},$$

and $\frac{bd}{8}$ is one quarter of $\frac{bd}{2}$.

(d) Take any triangle and any one of its sides. Slide and turn the triangle so that side starts at the origin and runs along the positive x-axis, and flip it over the axis if the third corner is below. Moving a triangle changes neither its area nor where its midpoints sit on it. Every triangle then has corners $(0, 0)$, $(b, 0)$, (c, d) with $b > 0$ and $d > 0$, so the letters of (c) cover every triangle at once. The algebra did not test more triangles; it covered all of them, which is what makes it a proof.

7. Claim: if the three sides of a triangle have whole-number lengths, then its area is a whole number.

(a) Mateo tests three triangles: a right triangle with legs 3 and 4 and hypotenuse 5, a right triangle with legs 6 and 8 and hypotenuse 10, and the triangle with sides 5, 5 and 6. Find each area.

(b) Show exactly that the triangle with all three sides of length 2 is a counterexample.

(c) Mateo repairs the claim: if the sides of a triangle are whole numbers and not all equal, then its area is a whole number. Test the repair on the triangle with sides 2, 2 and 3, and say what was wrong with this way of repairing.

ANSWER

(a) 6, 24 and 12. (b) Its area is $\sqrt{3}$, which is not a whole number. (c) Its area is $\frac{3\sqrt{7}}{4}$, not a whole number, so the repair is false. It ruled out only the kind of triangle that broke the claim, not the reason the claim fails.

SOLUTION

(a) The right triangles have areas $\frac{1}{2}(3)(4) = 6$ and $\frac{1}{2}(6)(8) = 24$. Place the third triangle at (0, 0), (6, 0), (3, 4): its sides are 6, $\sqrt{3^2 + 4^2} = 5$ and 5, and its area is $\frac{1}{2}(6)(4) = 12$. All three agree with the claim.

(b) Place the triangle at (0, 0), (2, 0), $(1, \sqrt{3})$. Its sides are 2, $\sqrt{1^2 + 3} = 2$ and 2, so its sides are whole numbers: the hypothesis is met. Its area is $\frac{1}{2}(2)(\sqrt{3}) = \sqrt{3}$. Since $1 < 3 < 4$, $1 < \sqrt{3} < 2$, so the area is not

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a whole number. The triangle is a **counterexample**.

(c) Place the triangle at (0, 0), (3, 0), $\left(\frac{3}{2}, \frac{\sqrt{7}}{2}\right)$. Its sides are 3 and $\sqrt{\frac{9}{4}+\frac{7}{4}} = 2$ twice, so they are whole numbers and not all equal: the repair's hypothesis is met. Its area is $\frac{1}{2}(3)\left(\frac{\sqrt{7}}{2}\right) = \frac{3\sqrt{7}}{4}$. Its square is $\frac{63}{16}$, which is not the square of a whole number, so the area is not a whole number and the repair is **false**.

Mateo excluded only the kind of figure he had found. The equilateral triangle failed because its height is a square root, and many other triangles have such heights. A repair that shuts out one counterexample leaves every figure like it still in the claim.

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