Geometry Unit 1 worksheet: Lines and planes in space (answer key)

ANSWERS AT A GLANCE

1. (a) (−2, −4) (b) (7, 3) (c) the same line: every point of $y = -3x + 4$ (d) no point: the lines are parallel (e) (−2, 2) (f) (4, −3)
2. (a) 460 cm, a length (dimension 1) (b) 7400 cm^2, an area (dimension 2) (c) 42000 cm^3, a volume (dimension 3)
3. 26 points
4. 6 m
5. (a) always (b) sometimes (c) sometimes (d) always (e) sometimes (f) never
6. Intersect: $\overline{BC}$, $\overline{FA}$, $\overline{AA'}$, $\overline{BB'}$, $\overline{CD}$, $\overline{EF}$ (six). Parallel: $\overline{DE}$, $\overline{A'B'}$, $\overline{D'E'}$ (three). Skew: $\overline{B'C'}$, $\overline{C'D'}$, $\overline{E'F'}$, $\overline{F'A'}$, $\overline{CC'}$, $\overline{DD'}$, $\overline{EE'}$, $\overline{FF'}$ (eight).
7. (a) If it did, B would be in both planes, so on ℓ, which it is not. (b) $\overleftrightarrow{AB}$ meets P only at A, and ℓ lies in P and misses A, so by the piercing test the lines are skew.

WORKED SOLUTIONS

1. Find every point the two lines share. If they share no point, or are the same line, say so and explain how you know.

(a) $y = 3x + 2$ and $y = -x - 6$ (b) $x + y = 10$ and $y = x - 4$ (c) $y = -3x + 4$ and $6x + 2y = 8$

(d) $2x - y = 5$ and $y = 2x + 1$ (e) $y = \frac{1}{2}x + 3$ and $y = -2x - 2$ (f) $x = 4$ and $y = -2x + 5$

ANSWER

(a) (−2, −4) (b) (7, 3) (c) the same line: every point of $y = -3x + 4$ (d) no point: the lines are parallel (e) (−2, 2) (f) (4, −3)

SOLUTION

(a) Set the two expressions for y equal: $3x + 2 = -x - 6$, so $4x = -8$ and $x = -2$. Then $y = 3(-2) + 2 = -4$. Check: −(−2) − 6 = −4. The point is (−2, −4).

(b) Put $y = x - 4$ into the first equation: $x + (x - 4) = 10$, so $2x = 14$ and $x = 7$. Then $y = 7 - 4 = 3$. Check: 7 + 3 = 10. The point is (7, 3).

(c) Solve the second equation for y: $2y = -6x + 8$, so $y = -3x + 4$. That is the first line. The two equations describe one line, so the lines share every point of $y = -3x + 4$.

(d) Solve the first equation for y: $y = 2x - 5$. Both lines have slope 2 but different y-intercepts, −5 and 1, so they lie in one plane and never meet: they are parallel and share no point.

(e) $\frac{1}{2}x + 3 = -2x - 2$, so $\frac{5}{2}x = -5$ and $x = -2$. Then $y = \frac{1}{2}(-2) + 3 = 2$. Check: −2(−2) − 2 = 2. The point is (−2, 2).

(f) Every point of the first line has $x = 4$, so put $x = 4$ into the second: $y = -8 + 5 = -3$. The point is (4, −3). The lines are different and not parallel (one is upright, the other is not), so there is no second point.

2. A fish tank is a glass box with no lid. It is 40 cm long, 35 cm wide and 40 cm tall. A plastic strip covers every edge of the tank, including the four edges around the open top, and water fills it to a depth of 30 cm.

Find (a) the length of the strip, (b) the area of the glass, and (c) the volume of the water. For each, give the dimension of the figure being measured.

ANSWER

(a) 460 cm, a length (dimension 1) (b) 7400 cm^2, an area (dimension 2) (c) 42000 cm^3, a volume (dimension 3)

SOLUTION

(a) The strip lies along the 12 edges of the box, which come in fours, one four for each length. Edges have dimension 1, so this is a length:

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$$4(40 + 35 + 40) = 4(115) = 460 \text{ cm}$$

(b) The glass is the surface of the box without its top: the bottom, 40 × 35 = 1400, the front and back, 2(40 × 40) = 3200, and the two ends, 2(35 × 40) = 2800. A surface has dimension 2, so this is an area:

$$1400 + 3200 + 2800 = 7400 \text{ cm}^2$$

(c) The water is a solid box 40 cm by 35 cm by 30 cm, not as tall as the tank. A solid has dimension 3, so this is a volume:

$$40 \times 35 \times 30 = 42000 \text{ cm}^3$$

3. On a city map every street is drawn as a straight line. 3 streets run north to south, 2 run east to west, and 4 run diagonally from southwest to northeast. Streets that run the same way are parallel, and no three streets pass through one point. At how many points do two streets cross?

ANSWER
26 points

SOLUTION
All the streets lie in one plane, the map. Two streets that run different ways are different lines that are not parallel, so they meet, and two different lines meet at most once: they cross exactly once. Two streets that run the same way are parallel and never meet.

No three streets pass through one point, so every crossing point belongs to exactly one pair of streets, and the number of points is the number of pairs that run different ways:

- north-south with east-west: 3 × 2 = 6
- east-west with diagonal: 2 × 4 = 8
- diagonal with north-south: 4 × 3 = 12

In all, 6 + 8 + 12 = 26 points.

4. A straight escalator runs from a point A on a balcony, 4 m above the ground floor of a mall, down through an opening in the floor to a point B in the basement. Treat the ground floor as a plane. The escalator is 20 m long and passes through the floor 8 m from A. How far below the ground floor is B?

ANSWER
6 m

SOLUTION
A is above the floor and B below it, so by space separation $\overline{AB}$ meets the floor, at a point X with $AX = 8$ m. Then $XB = 20 - 8 = 12$ m.

Along the segment the height changes at a steady rate, so the two pieces are in the same ratio as the heights on each side of the floor. If B is h m below the floor,

$$\frac{h}{4} = \frac{XB}{AX} = \frac{12}{8}, \qquad h = 4 \times \frac{12}{8} = 6$$

So B is 6 m below the ground floor. **Check.** 4 : 6 and 8 : 12 are the same ratio.

5. Decide whether each statement is always, sometimes or never true. For "sometimes", give one case where it is true and one where it is false.

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(a) Two different lines that meet at a point lie in one plane.

(b) Three different lines that all pass through one point lie in one plane.

(c) Two lines that are each parallel to the same plane are parallel to each other.

(d) Three different planes that share two different points share a whole line.

(e) Two lines that are each skew to a third line are skew to each other.

(f) A line that lies in one of two parallel planes meets a line that lies in the other.

ANSWER

(a) always (b) sometimes (c) sometimes (d) always (e) sometimes (f) never

SOLUTION

(a) **Always.** Call the meeting point X and take one more point on each line, Y and Z. These three points are not on one line, so exactly one plane holds them. Each line has two of its points in that plane, so the whole line lies in it.

(b) **Sometimes.** Three lines drawn on the floor through one spot lie in one plane, the floor. But the three edges of a room at a floor corner, where two walls meet the floor, pass through the corner and do not lie in one plane: the plane of the floor holds two of them, and the upright edge meets that plane only at the corner.

(c) **Sometimes.** Two lines drawn on a ceiling are both parallel to the floor. If they are drawn side by side and never meet, they are parallel; if they cross, they are not.

(d) **Always.** Call the points A and B. Any two of the planes are different planes that meet, so they share exactly $\overleftrightarrow{AB}$. Every one of the three planes therefore contains $\overleftrightarrow{AB}$, and all three share that whole line.

(e) **Sometimes.** In the box of the lesson (bottom $ABCD$, top $EFGH$), $\overleftrightarrow{AB}$ is skew to $\overleftrightarrow{CG}$ and to $\overleftrightarrow{EH}$, and those two are skew to each other: $\overleftrightarrow{CG}$ meets the top plane only at G, and $\overleftrightarrow{EH}$ lies in it and misses G. But $\overleftrightarrow{AB}$ is also skew to $\overleftrightarrow{DH}$, and $\overleftrightarrow{CG}$ and $\overleftrightarrow{DH}$ are parallel, not skew.

(f) **Never.** Parallel planes share no point. Every point of the first line is in one plane and every point of the second is in the other, so the lines share no point: they are parallel or skew.

6. A prism has a regular hexagon $ABCDEF$ for its bottom and a regular hexagon $A'B'C'D'E'F'$ for its top, with A' straight above A, B' above B, and so on. It has 18 edges. Sort the other 17 edges into three groups: those whose lines intersect $\overleftrightarrow{AB}$, those parallel to it, and those skew to it. Explain each group. (A sketch helps.)

ANSWER

Intersect: $\overline{BC}$, $\overline{FA}$, $\overline{AA'}$, $\overline{BB'}$, $\overline{CD}$, $\overline{EF}$ (six). Parallel: $\overline{DE}$, $\overline{A'B'}$, $\overline{D'E'}$ (three). Skew: $\overline{B'C'}$, $\overline{C'D'}$, $\overline{E'F'}$, $\overline{F'A'}$, $\overline{CC'}$, $\overline{DD'}$, $\overline{EE'}$, $\overline{FF'}$ (eight).

SOLUTION

Intersect. $\overline{BC}$, $\overline{FA}$, $\overline{AA'}$ and $\overline{BB'}$ share a corner with $\overline{AB}$. The edges $\overline{CD}$ and $\overline{EF}$ do not touch $\overline{AB}$, but their lines lie in the bottom plane with $\overleftrightarrow{AB}$ and are not parallel to it (in a regular hexagon only the opposite side $\overline{DE}$ is), so the lines meet outside the hexagon. Six in all.

Parallel. $\overline{DE}$ is the opposite side of the regular hexagon. $\overline{A'B'}$ is the opposite side of the rectangle $ABB'A'$. $\overline{D'E'}$ is parallel to $\overline{DE}$ (opposite sides of the rectangle $DEE'D'$), and two lines parallel to the same line are parallel, so it is parallel to $\overline{AB}$. Three in all.

Skew. The top edges $\overline{B'C'}$, $\overline{C'D'}$, $\overline{E'F'}$ and $\overline{F'A'}$ lie in the top plane, which is parallel to the bottom plane, so their lines are parallel or skew to $\overleftrightarrow{AB}$. Each is parallel to the bottom edge below it ($\overline{BC}$, $\overline{CD}$, $\overline{EF}$, $\overline{FA}$), and none of those is parallel to $\overline{AB}$, so none of the four is either: they are skew. The upright edges $\overline{CC'}$,

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$\overline{DD'}$, $\overline{EE'}$ and $\overline{FF'}$ each meet the bottom plane only at their lower end, and $\overleftrightarrow{AB}$ lies in that plane and misses C, D, E and F; by the piercing test they are skew. Eight in all.

Check: 6 + 3 + 8 = 17.

7. Planes P and Q meet in line ℓ. Point A is in P but not on ℓ, and point B is in Q but not on ℓ.

(a) Explain why $\overleftrightarrow{AB}$ does not lie in P.

(b) Explain why $\overleftrightarrow{AB}$ and ℓ are skew.

ANSWER

(a) If it did, B would be in both planes, so on ℓ, which it is not. (b) $\overleftrightarrow{AB}$ meets P only at A, and ℓ lies in P and misses A, so by the piercing test the lines are skew.

SOLUTION

First, A and B are different points: if they were the same point, it would be in P and in Q, so on ℓ, and A is not. So $\overleftrightarrow{AB}$ is a line.

(a) Suppose $\overleftrightarrow{AB}$ lay in P. Then B would be in P. It is also in Q, so it would be in $P \cap Q = \ell$. But B is not on ℓ. So $\overleftrightarrow{AB}$ does not lie in P.

(b) A line that does not lie in a plane meets it in at most one point. $\overleftrightarrow{AB}$ meets P at A, so A is its only point in P. The line ℓ lies in P and does not pass through A. By the piercing test, $\overleftrightarrow{AB}$ and ℓ are skew.

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