Geometry Unit 1 worksheet: Converses and biconditional definitions (answer key)

ANSWERS AT A GLANCE

1. (a) right angle, forward (b) congruent segments, backward (c) complementary angles, backward (d) angle bisector, forward
2. (a) 13 m (b) the converse of the Pythagorean theorem (c) the Pythagorean theorem
3. (a) Pythagorean theorem, the theorem (b) Pythagorean theorem, the converse (c) parallel lines, the converse (d) parallel lines, the contrapositive
4. (a) If two angles are congruent, then they are right angles. False: two $40°$ angles. (b) If $AM = \frac{1}{2}AB$, then M is the midpoint. False: $A = (0, 0)$, $B = (8, 0)$, $M = (0, 4)$. (c) If an angle is congruent to its supplement, then it is a right angle. True: $x = 180 - x$ gives $x = 90$.
5. (a) $OP = \sqrt{25 + 144} = 13$. (b) If $OP = 13$, then $x = 5$ and $y = 12$. False: $P = (13, 0)$. (c) False: $P = (5, -12)$. (d) Add $y > 0$: then $y^2 = 144$ forces $y = 12$.
6. (a) Too loose: sides 5, 5, 8. Repair: all three of its sides are congruent. (b) Too tight: $(0, 0)$, $(1, 1)$, $(2, 2)$. Repair: they all lie on one line. (c) Both: on a number line, $[-1, 1]$ and $[-3, 3]$ are let in; $[0, 2]$ and $[5, 7]$ are turned away. Repair: they have the same length.

WORKED SOLUTIONS

1. Each step below cites a definition as its reason. Name the definition, and say whether it runs **forward** (it unpacks a word you were handed into a fact) or **backward** (it earns a word once the condition is checked).

(a) $\angle K$ is a right angle, so $m\angle K = 90°$.

(b) $PQ = RS$, so $\overline{PQ} \cong \overline{RS}$.

(c) $m\angle 1 + m\angle 2 = 90°$, so $\angle 1$ and $\angle 2$ are complementary.

(d) $\overrightarrow{OX}$ bisects $\angle POQ$, so $\angle POX \cong \angle XOQ$.

ANSWER

(a) right angle, forward (b) congruent segments, backward (c) complementary angles, backward (d) angle bisector, forward

SOLUTION

Ask where the step starts. If it starts from the word and ends with the condition, the definition is unpacking the word: forward. If it starts from the condition and ends with the word, it is earning the word: backward.

(a) Starts from "right angle", ends with the measure $90°$. Definition of right angle, **forward**.

(b) Starts from equal lengths, ends with "congruent". Definition of congruent segments, **backward**.

(c) Starts from a sum of $90°$, ends with "complementary". Definition of complementary angles, **backward**.

(d) Starts from "bisects", ends with the two congruent angles. Definition of angle bisector, **forward**.

The same definition can serve in either direction; here only the starting point decides.

2. Grace is laying out a new rectangular plot in a community garden with string lines and stakes. The corner where the two string lines meet must be a right angle. From the corner stake she measures 5 m along the first string and 12 m along the second string, and ties a marker at each spot.

(a) If the corner is a right angle, how far apart are the two markers?

(b) She swings the second string until the markers are exactly that far apart. Which statement tells her the corner is now a right angle: the Pythagorean theorem or its converse?

(c) Which of the two did you use in (a)?

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ANSWER
(a) 13 m (b) the converse of the Pythagorean theorem (c) the Pythagorean theorem

SOLUTION
(a) If the corner is a right angle, the corner stake and the two markers make a right triangle with legs 5 m and 12 m, and the distance d between the markers is its hypotenuse. By the Pythagorean theorem,

$$d^2 = 5^2 + 12^2 = 25 + 144 = 169,$$

so $d = 13$ m.

(b) Now she knows the three sides, 5, 12 and 13, with $5^2 + 12^2 = 13^2$, and wants the angle. That runs from the sides to the right angle: the **converse** of the Pythagorean theorem. The angle opposite the side of 13 m, which is the corner, is a right angle.

(c) Part (a) started from the right angle and found a side: the **theorem** itself. The theorem alone could not check her corner, since it assumes the right angle she wants to make sure of.

3. Here are two true theorems.

- **Pythagorean theorem.** In $\triangle ABC$, if $m\angle C = 90^\circ$, then $a^2 + b^2 = c^2$.
- **Parallel lines.** If two lines cut by a transversal are parallel, then their corresponding angles are congruent.

Each conclusion below is correct. Say which theorem it rests on, and whether it uses the theorem itself, its converse, or its contrapositive.

(a) A bike frame's front triangle has a right angle where two tubes meet, and those tubes are 40 cm and 42 cm long. Isabel concludes the third tube is 58 cm long.

(b) A wall bracket is a triangle with sides 15 cm, 20 cm and 25 cm. Ben concludes it has a right angle.

(c) A straight access road crosses two rows of a vineyard. A pair of corresponding angles at the two crossings measure 63° and 63°. Sofia concludes the rows are parallel.

(d) At another pair of rows, the corresponding angles measure 63° and 61°. Sofia concludes these rows are not parallel.

ANSWER
(a) Pythagorean theorem, the theorem (b) Pythagorean theorem, the converse (c) parallel lines, the converse (d) parallel lines, the contrapositive

SOLUTION
Look at what is known and what is concluded, and match it to the if-part and the then-part.

(a) Known: a right angle. Concluded: a side length, from $40^2 + 42^2 = 1{,}600 + 1{,}764 = 3{,}364 = 58^2$. That runs from the if-part to the then-part of the **Pythagorean theorem itself**.

(b) Known: the three sides, with $15^2 + 20^2 = 225 + 400 = 625 = 25^2$. Concluded: a right angle. That runs from the then-part to the if-part: the **converse** of the Pythagorean theorem.

(c) Known: congruent corresponding angles. Concluded: parallel lines. The **converse** of the parallel lines theorem.

(d) Known: the corresponding angles are **not** congruent. Concluded: the lines are **not** parallel. If the rows were parallel the angles would be congruent, and they are not. That is the **contrapositive** of the parallel lines theorem, not its converse.

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4. Each statement is true. Write its converse, keeping the setting. Say whether the converse is true. If it is false, give a counterexample and show that it works; if it is true, show why.

(a) If two angles are right angles, then they are congruent.

(b) For points A, B and M in a plane: if M is the midpoint of $\overline{AB}$, then $AM = \frac{1}{2}AB$.

(c) For an angle with measure between 0° and 180°: if the angle is a right angle, then it is congruent to its supplement.

ANSWER

(a) If two angles are congruent, then they are right angles. False: two 40° angles. (b) If $AM = \frac{1}{2}AB$, then M is the midpoint. False: $A = (0,0)$, $B = (8,0)$, $M = (0,4)$. (c) If an angle is congruent to its supplement, then it is a right angle. True: $x = 180 - x$ gives $x = 90$.

SOLUTION

A counterexample to a converse meets the converse's if-part (the old conclusion) and fails its then-part (the old hypothesis).

(a) Converse: if two angles are congruent, then they are right angles. Two angles of 40° each are congruent, and neither is a right angle. **False.**

(b) Converse: if $AM = \frac{1}{2}AB$, then M is the midpoint of $\overline{AB}$. Take $A = (0,0)$, $B = (8,0)$ and $M = (0,4)$. Then $AB = 8$ and $AM = \sqrt{0^2 + 4^2} = 4 = \frac{1}{2} \cdot 8$, but M is not on $\overline{AB}$, so it is not the midpoint. **False.**

(c) Converse: if an angle is congruent to its supplement, then it is a right angle. Let the angle measure x°. Its supplement measures $(180 - x)^\circ$, and congruent angles have equal measures, so

$$x = 180 - x, \qquad 2x = 180, \qquad x = 90.$$

The angle measures 90°, so it is a right angle. **True.** Together with the statement, an angle is a right angle if and only if it is congruent to its supplement.

5. Let $O = (0,0)$, and let $P = (x,y)$ be a point. Take the statement

T. If $x = 5$ and $y = 12$, then $OP = 13$.

(a) Show that T is true.

(b) Write the **full converse** of T, which swaps the whole hypothesis with the conclusion. Is it true?

(c) Now swap only one part: "If $x = 5$ and $OP = 13$, then $y = 12$." Is this partial converse true?

(d) Add one condition about y to the hypothesis of the statement in (c) so that it becomes true, and show that it is.

ANSWER

(a) $OP = \sqrt{25 + 144} = 13$. (b) If $OP = 13$, then $x = 5$ and $y = 12$. False: $P = (13,0)$. (c) False: $P = (5,-12)$. (d) Add $y > 0$: then $y^2 = 144$ forces $y = 12$.

SOLUTION

(a) By the distance formula, $OP = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$.

(b) Full converse: if $OP = 13$, then $x = 5$ and $y = 12$. The point $P = (13,0)$ has $OP = \sqrt{13^2 + 0^2} = 13$, but $x = 13 \neq 5$. **False.**

(c) The point $P = (5,-12)$ has $x = 5$ and $OP = \sqrt{25 + 144} = 13$, but $y = -12 \neq 12$. **False.**

(d) Add "$y > 0$": if $x = 5$, $y > 0$ and $OP = 13$, then $y = 12$. Proof: $OP = 13$ means $x^2 + y^2 = 169$. With $x = 5$, $25 + y^2 = 169$, so $y^2 = 144$ and $y = 12$ or $y = -12$. Since $y > 0$, $y = 12$.

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The statement in (c) and the full converse are different statements, and here both are false while T is true. When a hypothesis has several parts, say which converse you mean.

6. Each sentence proposes a definition of a word you already know. Test both directions. If the proposal fails, give a counterexample for each direction that fails, say whether it is too loose, too tight or both, and repair it.

(a) "A triangle is equilateral if two of its sides are congruent."

(b) "Points are collinear if they all lie on the x-axis."

(c) "Two segments are congruent if they have the same midpoint."

ANSWER

(a) Too loose: sides 5, 5, 8. Repair: all three of its sides are congruent. (b) Too tight: (0, 0), (1, 1), (2, 2). Repair: they all lie on one line. (c) Both: on a number line, $[-1, 1]$ and $[-3, 3]$ are let in; $[0, 2]$ and $[5, 7]$ are turned away. Repair: they have the same length.

SOLUTION

Forward asks: does every real example meet the condition? If not, the proposal is too tight. Backward asks: is everything that meets the condition a real example? If not, it is too loose.

(a) Forward: an equilateral triangle has all three sides congruent, so certainly two of them. This holds. Backward: a triangle with sides 5, 5 and 8 exists, since 5 + 5 = 10 > 8, and it has two congruent sides, but 8 ≠ 5, so it is not equilateral. **Too loose.** Repair: "A triangle is equilateral if all three of its sides are congruent."

(b) Backward: points on the x-axis lie on one line, so they are collinear. This holds. Forward: (0, 0), (1, 1) and (2, 2) all lie on the line $y = x$, so they are collinear, but (1, 1) is not on the x-axis. **Too tight.** Repair: "Points are collinear if they all lie on one line."

(c) Backward: on a number line, the segments from −1 to 1 and from −3 to 3 both have midpoint 0, but their lengths are 2 and 6. Forward: the segments from 0 to 2 and from 5 to 7 both have length 2, so they are congruent, but their midpoints are 1 and 6. **Both.** Repair: "Two segments are congruent if they have the same length."

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