Geometry Unit 1 worksheet: Axioms, definitions and theorems (answer key)

ANSWERS AT A GLANCE

1. (a) neat, length 10 (b) not neat, length $\sqrt{13}$ (c) neat, length 4 (d) the length is $5n$, a whole number, so the segment is neat.
2. (a) $56 = 7^2 + 7$ is a near-square; 40 is not. (b) $4N + 1 = (2k + 1)^2$ (c) $N = 2(2a^2 + a)$ when $k = 2a$ and $N = 2(2a^2 + 3a + 1)$ when $k = 2a + 1$.
3. (a) 6 lines (b) 26 lines
4. (a) For example, four points A, B, C, D and the six lines $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C\}$, $\{B, D\}$, $\{C, D\}$. (b) For example, six points 1 to 6 and the two lines $\{1, 2, 3\}$ and $\{4, 5, 6\}$.
5. (a) Results 2, 4 and 5 lose their proofs; Results 1, 3 and 6 keep theirs. (b) a corollary of Result 4 (c) it makes a circle: Result 5 stands on Result 4, which stands on Result 3.
6. (a) $\{2, 6, 7\}$ and $\{2, 4, 9\}$ (b) $\{4, 5, 6\}$, $\{2, 5, 8\}$, $\{1, 5, 9\}$, $\{3, 5, 7\}$: four lines, as forced. (c) for example $\{1, 2, 3\}$ and $\{4, 5, 6\}$; the statement cannot be proved from S1 and S2.
7. See the proof below.

WORKED SOLUTIONS

1. Call a segment in the coordinate plane **neat** if its length is a whole number.

(a) Use the distance formula to decide whether the segment from (−2, 3) to (4, −5) is neat.

(b) Decide whether the segment from (0, 0) to (2, 3) is neat.

(c) Decide whether the segment from (0, 0) to $(\sqrt{7}, 3)$ is neat.

(d) Show that for every positive integer n, the segment from (n, n) to $(4n, 5n)$ is neat, and give its length.

ANSWER

(a) neat, length 10 (b) not neat, length $\sqrt{13}$ (c) neat, length 4 (d) the length is $5n$, a whole number, so the segment is neat.

SOLUTION

The test: find the length, then ask whether it is a whole number.

(a) $\sqrt{(4 - (-2))^2 + (-5 - 3)^2} = \sqrt{36 + 64} = \sqrt{100} = 10$, a whole number, so the segment is neat.

(b) $\sqrt{2^2 + 3^2} = \sqrt{13}$. Since $3^2 = 9 < 13 < 16 = 4^2$, the length lies strictly between 3 and 4, so it is not a whole number. The segment is not neat.

(c) $\sqrt{(\sqrt{7})^2 + 3^2} = \sqrt{7 + 9} = \sqrt{16} = 4$, so the segment is neat. The definition asks only about the length. It says nothing about the endpoints, so an endpoint whose coordinate is not a whole number does not stop the segment from passing.

(d) To fold, we must show that the length is a whole number. The coordinates change by $4n - n = 3n$ across and $5n - n = 4n$ up, so by the distance formula the length is

$$\sqrt{(3n)^2 + (4n)^2} = \sqrt{9n^2 + 16n^2} = \sqrt{25n^2} = 5n$$

where $\sqrt{25n^2} = 5n$ because n is positive. Since n is a positive integer, $5n$ is a whole number. So the segment meets the condition of the definition, and it is neat.

Check. For $n = 2$ the segment runs from (2, 2) to (8, 10), and $\sqrt{36 + 64} = 10 = 5 \cdot 2$.

2. Call an integer a **near-square** if it equals $k^2 + k$ for some integer k.

(a) Decide whether 56 is a near-square and whether 40 is a near-square.

(b) Show that if N is a near-square, then $4N + 1$ is the square of an integer.

(c) Show that every near-square is even. (Recall: an integer is odd if it equals $2m + 1$ for some integer m, and every integer is even or odd.)

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ANSWER
(a) $56 = 7^2 + 7$ is a near-square; 40 is not. (b) $4N + 1 = (2k + 1)^2$ (c) $N = 2(2a^2 + a)$ when $k = 2a$ and $N = 2(2a^2 + 3a + 1)$ when $k = 2a + 1$.

SOLUTION
(a) $56 = 49 + 7 = 7^2 + 7$, with $k = 7$ an integer, so 56 is a near-square.

For 40, suppose $k^2 + k = 40$. Then $k^2 + k - 40 = 0$, and the quadratic formula gives

$$k = \frac{-1 \pm \sqrt{1 + 160}}{2} = \frac{-1 \pm \sqrt{161}}{2}$$

Since $12^2 = 144 < 161 < 169 = 13^2$, $\sqrt{161}$ is not a whole number, so neither value of k is an integer. The equation leaves no other choice, so 40 is not a near-square.

(b) Unfold: $N = k^2 + k$ for some integer k. Then

$$4N + 1 = 4k^2 + 4k + 1 = (2k + 1)^2$$

and $2k + 1$ is an integer, so $4N + 1$ is the square of an integer. Check: $4 \cdot 56 + 1 = 225 = 15^2$, and $2 \cdot 7 + 1 = 15$.

(c) Unfold: $N = k^2 + k$ for some integer k. The integer k is even or odd.

If k is even, $k = 2a$ for some integer a, and $N = 4a^2 + 2a = 2(2a^2 + a)$.

If k is odd, $k = 2a + 1$ for some integer a, and $N = (4a^2 + 4a + 1) + (2a + 1) = 4a^2 + 6a + 2 = 2(2a^2 + 3a + 1)$.

In each case N is 2 times an integer, so by the definition of even, N is even.

3. A system has two axioms about points and lines.

S1. Every line has exactly three points on it.

S2. Any two different points lie on exactly one line.

A world obeys S1 and S2 and has exactly 13 points.

(a) How many lines pass through a given point P?

(b) How many lines does the world have?

ANSWER
(a) 6 lines (b) 26 lines

SOLUTION
(a) Each of the other 12 points lies on exactly one line with P (S2). Each line through P holds exactly two points besides P (S1). So the lines through P split the other 12 points into groups of 2, and there are $12 \div 2 = 6$ of them.

(b) Count pairs of points. There are $\binom{13}{2} = \frac{13 \cdot 12}{2} = 78$ pairs, and each lies on exactly one line (S2). A line with three points holds $\binom{3}{2} = 3$ pairs (S1). So there are $78 \div 3 = 26$ lines.

Check. Each of the 26 lines has 3 points, and each of the 13 points is on 6 lines: $26 \cdot 3 = 78 = 13 \cdot 6$.

4. Add a third axiom to S1 and S2.

S1. Every line has exactly three points on it.

S2. Any two different points lie on exactly one line.

S3. There are at least two lines.

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(a) Show that S1 cannot be proved from S2 and S3: describe a world, giving its points and its lines, in which S2 and S3 are true and S1 is false.

(b) Show in the same way that S2 cannot be proved from S1 and S3.

ANSWER

(a) For example, four points A, B, C, D and the six lines $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C\}$, $\{B, D\}$, $\{C, D\}$. (b) For example, six points 1 to 6 and the two lines $\{1, 2, 3\}$ and $\{4, 5, 6\}$.

SOLUTION

(a) Take four points A, B, C, D, and one line for each pair: $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C\}$, $\{B, D\}$, $\{C, D\}$. Any two different points lie on the line made of exactly those two, and on no other, so S2 holds. There are six lines, so S3 holds. Every line has two points, not three, so S1 fails. A world where the other axioms hold and S1 fails means no proof of S1 from S2 and S3 exists.

(b) Take six points 1 to 6 and the two lines $\{1, 2, 3\}$ and $\{4, 5, 6\}$. Each line has exactly three points, so S1 holds, and there are two lines, so S3 holds. The points 1 and 4 lie on no line at all, so S2 fails. So no proof of S2 from S1 and S3 exists.

Other worlds work too. A world is correct if it can be checked against each axiom in the same way.

5. A short text proves six results. Each line says what its proof uses.

- Result 1 is proved from Axioms A and B.
- Result 2 is proved from Result 1 and a definition.
- Result 3 is proved from Axiom C alone.
- Result 4 is proved from Results 2 and 3.
- Result 5 is proved from Result 4 in two lines.
- Result 6 is proved from Results 1 and 3.

(a) A reader finds a gap in the proof of Result 2. Which results are left without a proof until the gap is mended, and which keep their proofs?

(b) Which of the words theorem, lemma or corollary best suits Result 5, and of which result?

(c) The author wants to rewrite the proof of Result 3 so that it cites Result 5. Explain why that is not allowed.

ANSWER

(a) Results 2, 4 and 5 lose their proofs; Results 1, 3 and 6 keep theirs. (b) a corollary of Result 4 (c) it makes a circle: Result 5 stands on Result 4, which stands on Result 3.

SOLUTION

(a) A flaw travels upward, to every result that stands on the flawed one. Result 2 itself is unproved. Result 4 uses Result 2, so it is unproved. Result 5 uses Result 4, so it is unproved. Nothing else uses Results 2, 4 or 5. Result 1 stands beneath Result 2 and never used it, Result 3 uses only Axiom C, and Result 6 uses only Results 1 and 3. Those three keep their proofs.

(b) Result 5 follows from Result 4 in two lines, a quick consequence of a result just proved, so it is a corollary of Result 4.

(c) Follow the arrows: Result 5 is proved from Result 4, and Result 4 from Result 3. If the proof of Result 3 cited Result 5, then Result 3 would be used to prove Result 5 and Result 5 to prove Result 3. Neither would be proved before the other. That is a circle, and a circle proves nothing.

6. Here is a world for the axioms S1 and S2 of the previous problem. It has nine points, named 1 to 9, and these twelve lines:

$\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9\}$, $\{1, 4, 7\}$, $\{2, 5, 8\}$, $\{3, 6, 9\}$, $\{1, 5, 9\}$, $\{2, 6, 7\}$, $\{3, 4, 8\}$, $\{3, 5, 7\}$, $\{1, 6, 8\}$, $\{2, 4, 9\}$

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(a) Name the line through the points 2 and 6, and the line through the points 4 and 9.

(b) List the lines through the point 5. Does their number agree with what S1 and S2 force for a world with nine points?

(c) Find two different lines with no point in common. What does this show about the statement "any two different lines have a point in common", for a system whose only axioms are S1 and S2?

ANSWER

(a) $\{2, 6, 7\}$ and $\{2, 4, 9\}$ (b) $\{4, 5, 6\}$, $\{2, 5, 8\}$, $\{1, 5, 9\}$, $\{3, 5, 7\}$: four lines, as forced. (c) for example $\{1, 2, 3\}$ and $\{4, 5, 6\}$; the statement cannot be proved from S1 and S2.

SOLUTION

(a) Search the list for a line holding both points. The line through 2 and 6 is $\{2, 6, 7\}$, and the line through 4 and 9 is $\{2, 4, 9\}$. S2 promises there is only one of each.

(b) The lines containing 5 are $\{4, 5, 6\}$, $\{2, 5, 8\}$, $\{1, 5, 9\}$ and $\{3, 5, 7\}$, four lines. The axioms force this number: the other 8 points each lie on exactly one line with 5 (S2), and each such line holds two of them (S1), so there are $8 \div 2 = 4$ lines through 5. They agree.

(c) The lines $\{1, 2, 3\}$ and $\{4, 5, 6\}$ share no point (so do $\{1, 5, 9\}$ and $\{2, 6, 7\}$, among others). Every line has three points and every pair of points lies on exactly one line, so this is a world in which S1 and S2 are true and the statement is false. By "to show that a statement cannot be proved, build a world", no proof of the statement from S1 and S2 exists.

7. Take the angle sum theorem as already proved: the three angles of any triangle add to 180°. An angle is **obtuse** if its measure is greater than 90° and less than 180°.

(a) Prove that a triangle has at most one obtuse angle.

(b) Why is the result of (a) usually called a corollary rather than a theorem?

(c) The four angles of a convex quadrilateral add to 360°. Use the same kind of argument to prove that a convex quadrilateral cannot have four obtuse angles. Then explain why this argument does not show that a convex quadrilateral has at most one obtuse angle.

ANSWER

(a) Two obtuse angles would already add to more than 180°. (b) Its proof is one citation of the angle sum theorem and a line of arithmetic. (c) Four obtuse angles would add to more than 360°; two obtuse angles add to more than 180° but less than 360°, so no contradiction arises.

SOLUTION

(a) Suppose a triangle had two obtuse angles, α and β, and call its third angle γ. Then $\alpha > 90^\circ$ and $\beta > 90^\circ$, so $\alpha + \beta > 180^\circ$. The third angle of a triangle has a positive measure, so

$$\alpha + \beta + \gamma > 180^\circ$$

But the angle sum theorem says $\alpha + \beta + \gamma = 180^\circ$. The supposition is impossible, so a triangle has at most one obtuse angle. ■

(b) The proof cites one result just proved, the angle sum theorem, and adds a line of arithmetic. A quick consequence of a result, with its work already done, is what the word corollary names. It is no less certain for that: it is proved.

(c) Suppose all four angles of a convex quadrilateral were obtuse. Each is greater than 90°, so their sum is greater than $4 \cdot 90° = 360°$. But the four angles add to exactly 360°, so the supposition is impossible: a convex quadrilateral cannot have four obtuse angles. ■

The argument of (a) worked because two obtuse angles already overshoot the whole total of 180°. In a quadrilateral, two obtuse angles add to more than 180° but less than 360° (each is less than 180°), which

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> the total of 360° still has room for. No contradiction arises, so the argument proves nothing about "at most one". Only when four angles are all obtuse does the sum overshoot 360°.

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