Basis and dimension

A basis is a short list that builds every vector of a subspace, each in only one way. Every basis of the subspace has the same number of vectors, and that number is its dimension.

WORDS TO KNOW

Basis of V: a list of vectors in V that is linearly independent and spans V.

Standard basis of $\mathbb{R}^n$: $\mathbf{e}_1, \ldots, \mathbf{e}_n$. The coordinates of a vector in it are its entries.

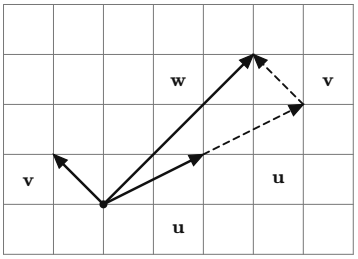
Coordinates: the weights $(c_1, \ldots, c_k)$ in $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$, in the order of the basis.

Dimension $\dim V$**:** the number of vectors in a basis of V.

KEY IDEAS

1. A basis builds every vector, each in one way

Spanning says every vector of V can be built. Independence says nothing in the list is wasted. A basis is like a set of axes for V, and coordinates are the steps along each axis.



Relative to a basis of V, every $\mathbf{x}$ in V has exactly one list of coordinates.

A subspace has many bases, and they give one vector different coordinates.

2. Every basis has the same size

Exchange bound: if V is spanned by m vectors, every independent list in V has at most m vectors.

So every basis of V has the same number of vectors, $\dim V$. Also $\dim \mathbb{R}^n = n$, and $\dim\{\mathbf{0}\} = 0$ (its basis is the empty list).

Every subspace of $\mathbb{R}^n$ has a basis, and its dimension is at most n.

3. When you know the dimension, two conditions are enough

Let $\dim V = k$. Then k independent vectors in V span V, and k vectors that span V are independent.

More than k vectors in V are dependent. Fewer than k do not span V.

If U is a subspace inside V, then $\dim U \leq \dim V$, and $\dim U = \dim V$ forces $U = V$.

4. Finding a basis by row reduction

Of a span: put the vectors in as columns and reduce to an echelon form. The original vectors whose columns hold pivots are a basis of their span.

Of the solutions of $A\mathbf{x} = \mathbf{0}$: write the general solution with one vector for each free variable. Those vectors are a basis, so the dimension is the number of free variables.

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WATCH OUT

- **Keep the original vectors.** The echelon columns usually span a different subspace. Use them only to see which original vectors to keep.
- **Dimension counts basis vectors, not entries.** A plane through the origin in $\mathbb{R}^4$ has dimension 2, though its vectors have 4 entries.
- **The counting shortcut needs both conditions.** The vectors must lie in V, and there must be exactly $\dim V$ of them.
- **Coordinates depend on the basis and its order.** Only the number of coordinates is fixed.

WORKED EXAMPLES

Example 1. Show that (1, 1, 0), (2, 3, 1), (1, 0, 2) is a basis of $\mathbb{R}^3$, and find the coordinates of (1, −1, 1).

1. Put the vectors in as columns, with (1, −1, 1) on the right. $R_2 - R_1$, then $R_3 - R_2$:
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & 3 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 3 & 3 \end{array}\right]$$
2. A pivot in every column makes the list independent. A pivot in every row makes it span $\mathbb{R}^3$. So it is a basis.
3. Solve from the bottom: $3c_3 = 3$ gives $c_3 = 1$. Then $c_2 - 1 = -2$ gives $c_2 = -1$. Then $c_1 + 2(-1) + 1 = 1$ gives $c_1 = 2$.

Answer: a basis, and the coordinates are (2, −1, 1). **Check:** $2(1,1,0) - (2,3,1) + (1,0,2) = (1,-1,1)$.

Example 2. Choose from $\mathbf{v}_1 = (1,0,1,3)$, $\mathbf{v}_2 = (0,1,2,1)$, $\mathbf{v}_3 = (2,-1,0,5)$, $\mathbf{v}_4 = (1,1,0,1)$ a basis of their span, and give its dimension.

1. Put the vectors in as columns. $R_3 - R_1$ and $R_4 - 3R_1$, then $R_3 - 2R_2$ and $R_4 - R_2$, then $R_4 - R_3$:
$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & 0 & 0 \\ 3 & 1 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
2. The pivots are in columns 1, 2 and 4. Keep the original $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_4$.
3. To see why $\mathbf{v}_3$ can go, put weight 1 on column 3 and solve: row 3 gives $c_4 = 0$, row 2 gives $c_2 = 1$, row 1 gives $c_1 = -2$. So $-2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$.

Answer: $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$ is a basis, and the dimension is 3. **Check:**
$2\mathbf{v}_1 - \mathbf{v}_2 = (2,0,2,6) - (0,1,2,1) = (2,-1,0,5) = \mathbf{v}_3$.

Example 3. Show that (1, 3, 1), (1, 1, −1) is a basis of the plane P given by $2x - y + z = 0$, without checking that they span. Find the coordinates of (3, 5, −1).

1. Both lie in P: 2 − 3 + 1 = 0 and 2 − 1 − 1 = 0. Neither is a multiple of the other, so they are independent.
2. Solving $y = 2x + z$ leaves x and z free, so $\dim P = 2$.
3. Two independent vectors in P, and $\dim P = 2$, so by idea 3 they span P. They are a basis.
4. Coordinates: $a(1,3,1) + b(1,1,-1) = (a+b,\ 3a+b,\ a-b)$. The first two entries give $a + b = 3$ and $3a + b = 5$, so $a = 1$ and $b = 2$.

Answer: a basis of P, and the coordinates of (3, 5, −1) are (1, 2). **Check:** the third entry is 1 − 2 = −1, as it should be.

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