Linear combination

Scale some vectors and add them. Going forward is arithmetic. Going backward, finding the amounts that give a target vector, means solving a system.

WORDS TO KNOW

Linear combination: a vector $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$, made by scaling vectors of one $\mathbb{R}^n$ and adding.

Weights: the scalars $c_1, \ldots, c_k$. They can be any real numbers: negative, fractions or zero.

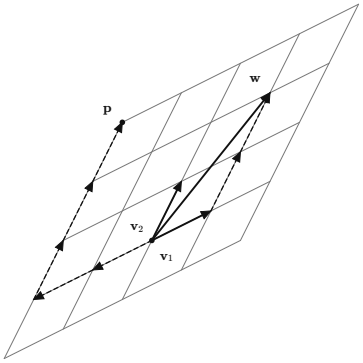
$\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$: some choice of weights makes the combination equal $\mathbf{b}$.

Standard vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$: $\mathbf{e}_i$ has 1 in position i and 0 everywhere else.

KEY IDEAS

1. Scale each vector, then add

In the plane, a combination is a walk: c_1 steps along $\mathbf{v}_1$, then c_2 steps along $\mathbf{v}_2$. A negative weight walks backward, and a fraction stops between grid lines. The weights are the point's address on the slanted grid.



$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

All weights 0 give $\mathbf{0}$, so $\mathbf{0}$ is a combination of any list. A sum, a difference and a multiple are combinations too.

2. Finding the weights means solving a system

Two vectors are equal when every position matches. So one vector equation becomes one number equation per position, and the unknowns are the weights.

$\mathbf{b} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ exactly when the system with augmented matrix $[\,\mathbf{v}_1 \;\cdots\; \mathbf{v}_k \mid \mathbf{b}\,]$ has a solution. Each solution is a list of weights.

Every $\mathbf{x}$ in $\mathbb{R}^n$ is $x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$, and its entries are the only weights that work.

3. One list of weights, many, or none

When one vector of the list is built from the others, the weights can shift. For example, if $\mathbf{v} = -2\mathbf{u}$, every combination of $\mathbf{u}$ and $\mathbf{v}$ is a multiple of $\mathbf{u}$. A vector on that line has infinitely many lists of weights, and a vector off it has none.

A combination of combinations is a combination: if $\mathbf{x}$ and $\mathbf{y}$ are combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, so is $r\mathbf{x} + s\mathbf{y}$.

WATCH OUT

- **Use every equation.** With more positions than weights, a few equations fix the weights. The rest must also hold, or there are no weights at all.
- **Use one $\mathbb{R}^n$.** All the vectors must have the same number of entries.
- **A real setting may limit the weights.** Liters of paint cannot be negative, and packs come whole. "Is it a combination?" allows every real weight. "Can it be mixed?" is a stricter question.
- **The answer belongs to the whole list.** Adding a vector to the list can turn "no" into "yes".

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WORKED EXAMPLES

Example 1. Let $\mathbf{v}_1 = (1, -2, 0, 3)$, $\mathbf{v}_2 = (2, 1, -1, 0)$ and $\mathbf{v}_3 = (0, 4, 2, -2)$. Find the combination with weights 3, −2 and $\frac{1}{2}$.

1. Scale each vector: $3\mathbf{v}_1 = (3, -6, 0, 9)$, $-2\mathbf{v}_2 = (-4, -2, 2, 0)$ and $\frac{1}{2}\mathbf{v}_3 = (0, 2, 1, -1)$.
2. Add position by position: (3 − 4 + 0, −6 − 2 + 2, 0 + 2 + 1, 9 + 0 − 1).

Answer: $3\mathbf{v}_1 - 2\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 = (-1, -6, 3, 8)$.

Example 2. Let $\mathbf{a}_1 = (1, 3, -2)$ and $\mathbf{a}_2 = (2, -1, 1)$. Is $\mathbf{b} = (4, 5, -3)$ a combination of them? Is $\mathbf{d} = (4, 5, 0)$?

1. For **b**, match entries: $c_1 + 2c_2 = 4$, $3c_1 - c_2 = 5$ and $-2c_1 + c_2 = -3$.
2. The first gives $c_1 = 4 - 2c_2$. Put it in the second: $12 - 7c_2 = 5$, so $c_2 = 1$ and $c_1 = 2$.
3. These are the only values the first two equations allow. Test the third: −2(2) + 1 = −3. It holds.
4. For **d**, the first two equations are the same, so again $c_1 = 2$, $c_2 = 1$. Now the third needs −3 = 0, which fails.

Answer: $\mathbf{b} = 2\mathbf{a}_1 + \mathbf{a}_2$, with only these weights. **d** is not a combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. **Check:** $(2, 6, -4) + (2, -1, 1) = (4, 5, -3)$.

Example 3. Find the weights that write any (x, y) as $c_1(2, 1) + c_2(1, -2)$. Use them for (0, 5) and for $\mathbf{e}_1$.

1. Match entries: $2c_1 + c_2 = x$ and $c_1 - 2c_2 = y$.
2. Twice the first plus the second leaves $5c_1 = 2x + y$. The first minus twice the second leaves $5c_2 = x - 2y$.
3. So $c_1 = \frac{2x+y}{5}$ and $c_2 = \frac{x-2y}{5}$. Weights exist for every (x, y), and only one list.
4. For (0, 5): $c_1 = 5 \div 5 = 1$ and $c_2 = -10 \div 5 = -2$. For $\mathbf{e}_1 = (1, 0)$: $c_1 = \frac{2}{5}$ and $c_2 = \frac{1}{5}$.

Answer: (0, 5) = (2, 1) − 2(1, −2) and $\mathbf{e}_1 = \frac{2}{5}(2, 1) + \frac{1}{5}(1, -2)$. **Check:** (2, 1) − (2, −4) = (0, 5).

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