Linear independence

Is any vector in a list wasted, because it can be built from the others? One system with right side zero answers it, and also tells you whether weights are unique.

WORDS TO KNOW

Trivial combination: all weights 0. It always gives $\mathbf{0}$.

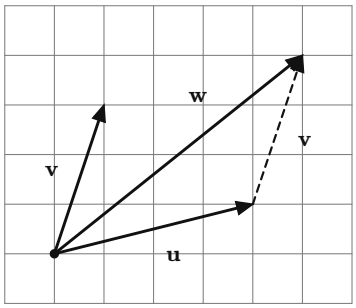
Linearly dependent: some weights, not all zero, give $\mathbf{0}$.

Linearly independent: the trivial combination is the only one that gives $\mathbf{0}$.

Dependence relation: an equation $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ with weights not all zero.

KEY IDEAS

1. Independent means only the trivial combination gives zero



Read a combination as a walk. A dependence relation is a walk that ends where it began without standing still.

> $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are independent when $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ only for $c_1 = \cdots = c_k = 0$.
>
> A list of two or more vectors is dependent exactly when at least one of its vectors is a linear combination of the others.

Independence belongs to the whole list, not to one vector in it. A list with a vector repeated is dependent, since $\mathbf{v} - \mathbf{v} = \mathbf{0}$.

2. Small lists at a glance

> A list containing $\mathbf{0}$ is dependent. One vector alone is independent exactly when it is not $\mathbf{0}$.
>
> Two vectors are dependent exactly when one is a multiple of the other.
>
> More than n vectors in $\mathbb{R}^n$ are always dependent.

3. The test: a pivot in every column

> Put the vectors in as the columns of a matrix and row reduce to an echelon form. The list is independent exactly when every column has a pivot.
>
> A column with no pivot gives a free weight. Set it to 1 and solve upward: that is a dependence relation.

The right side is $\mathbf{0}$ and row operations keep it $\mathbf{0}$, so you can leave it off.

4. Independent means unique weights

> A list is independent exactly when every combination of it has only one list of weights. For a dependent list, every combination has infinitely many.

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WATCH OUT

- **Dependent does not mean every vector is redundant.** It means at least one is a combination of the others. Another vector in the same list may not be.
- **Pivots in columns decide, not rows.** A zero row in the echelon form only says 0 = 0. An independent list can have one.
- **Few vectors prove nothing.** At most n vectors in $\mathbb{R}^n$ may be independent or dependent. Only more than n settles it without work.

WORKED EXAMPLES

Example 1. Are $\mathbf{a} = (1, 2, 0, 1)$, $\mathbf{b} = (2, 5, 1, 0)$ and $\mathbf{c} = (1, 0, 1, 2)$ linearly independent?

1. Put them in as columns. $R_2 - 2R_1$ and $R_4 - R_1$ give rows (0, 1, −2) and (0, −2, 1).
2. $R_3 - R_2$ and $R_4 + 2R_2$ give rows (0, 0, 3) and (0, 0, −3). Then $R_4 + R_3$ gives a zero row:
$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$
3. Every column has a pivot. The zero row does not matter.

Answer: independent. **Check:** from the bottom, $3c_3 = 0$ gives $c_3 = 0$, then $c_2 - 2c_3 = 0$ gives $c_2 = 0$, then $c_1 + 2c_2 + c_3 = 0$ gives $c_1 = 0$.

Example 2. Find every dependence relation among $\mathbf{v}_1 = (1, 2, -1)$, $\mathbf{v}_2 = (2, 3, 1)$, $\mathbf{v}_3 = (5, 9, -2)$.

1. Columns in, then $R_2 - 2R_1$ and $R_3 + R_1$, then $R_3 + 3R_2$:
$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 9 \\ -1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$
2. Column 3 has no pivot, so $c_3 = t$ is free. Row 2 gives $-c_2 - t = 0$, so $c_2 = -t$. Row 1 gives $c_1 = -2c_2 - 5c_3 = -3t$.
3. Every relation is $-3t\,\mathbf{v}_1 - t\,\mathbf{v}_2 + t\,\mathbf{v}_3 = \mathbf{0}$ with $t \neq 0$. Take $t = 1$ and solve for $\mathbf{v}_3$.

Answer: dependent, and $\mathbf{v}_3 = 3\mathbf{v}_1 + \mathbf{v}_2$. All three weights are nonzero, so each vector is a combination of the other two. **Check:** $(3, 6, -3) + (2, 3, 1) = (5, 9, -2)$.

Example 3. Decide without a system: (a) (4, 2, −6), (−10, −5, 15); (b) (1, 5, 2), (2, 10, 3); (c) (2, −1, 0), (0, 0, 0), (5, 1, 3).

1. (a) Two vectors: is one a multiple of the other? The first entries allow only the factor $-\frac{5}{2}$, and $-\frac{5}{2}(4, 2, -6) = (-10, -5, 15)$ in every entry.
2. (b) The first entries force the factor 2, but in the third entries $2 \cdot 2 = 4 \neq 3$. The reverse factor $\frac{1}{2}$ fails the same entry. Neither is a multiple of the other.
3. (c) The list contains **0**. Weight 1 on **0** and 0 on the rest gives **0**, with a nonzero weight.

Answer: (a) dependent, with $5(4, 2, -6) + 2(-10, -5, 15) = \mathbf{0}$. (b) independent. (c) dependent.

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