Span

Everything you can reach from a list of vectors by scaling and adding is their span. What shape is it, is a given vector in it, and is it all of space?

WORDS TO KNOW

Span: $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

Span a set: the vectors span W (for example all of $\mathbb{R}^m$) when their span is exactly W.

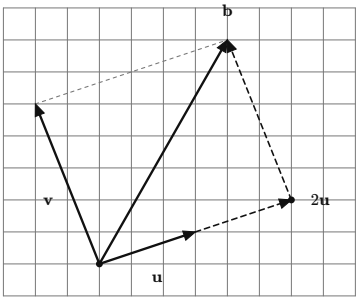
In the span: $\mathbf{b}$ is in the span when it is a linear combination of the list.

Pivot: the first nonzero entry of a row in an echelon form.

KEY IDEAS

1. A span is flat and passes through the origin

The span always holds $\mathbf{0}$ (all weights 0) and each $\mathbf{v}_i$. The list is finite, but the span is usually an infinite set.



$$\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \{\, c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k : c_1, \ldots, c_k \text{ real} \,\}$$

One nonzero vector spans a line through the origin. Two vectors, neither a multiple of the other, span a plane through the origin. In $\mathbb{R}^2$ that plane is all of $\mathbb{R}^2$.

2. Test a vector, or find the equations of the span

$\mathbf{b}$ is in $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ exactly when the system with augmented matrix $[\,\mathbf{v}_1 \;\cdots\; \mathbf{v}_k \mid \mathbf{b}\,]$ is consistent.

To describe the whole span, row reduce with a general right side (x, y, z). Each row that becomes zero on the left gives an equation the right side must satisfy. In $\mathbb{R}^3$ a plane through the origin takes one equation, and a line takes two.

3. Spanning all of $\mathbb{R}^m$: a pivot in every row

Put $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^m$ as the columns of a matrix A. They span $\mathbb{R}^m$ exactly when an echelon form of A has a pivot in every row.

Each column holds at most one pivot, so fewer than m vectors never span $\mathbb{R}^m$.

4. A vector already in the span adds nothing

If $\mathbf{w}$ is in $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, then adding $\mathbf{w}$ to the list leaves the span the same. A vector outside the span makes it bigger.

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WATCH OUT

- **Counting vectors is not enough.** Two vectors may span only a line, if one is a multiple of the other. Three vectors in $\mathbb{R}^3$ may span only a plane.
- **Count pivots in rows, not columns.** A pivot in every row is spanning. A pivot in every column is a different property.
- **A span always passes through the origin.** So a line or plane that misses $\mathbf{0}$, such as $y = x + 1$, is never a span.
- **Every real weight is allowed.** Negative weights and fractions count. A question that allows only nonnegative weights asks something stricter.

WORKED EXAMPLES

Example 1. Let $\mathbf{v}_1 = (1, 1, 2)$ and $\mathbf{v}_2 = (2, 3, 1)$. Find an equation for their span, and decide whether $\mathbf{b} = (1, 4, -7)$ and $\mathbf{d} = (2, 1, 3)$ are in it.

1. Row reduce with a general right side. $R_2 - R_1$ and $R_3 - 2R_1$ give rows $[\,0 \;\; 1 \mid y - x\,]$ and $[\,0 \;\; -3 \mid z - 2x\,]$. Then $R_3 + 3R_2$:
$$\left[\begin{array}{cc|c} 1 & 2 & x \\ 0 & 1 & y - x \\ 0 & 0 & -5x + 3y + z \end{array}\right]$$
2. The system is consistent exactly when $-5x + 3y + z = 0$. So the span is the plane $5x - 3y - z = 0$.
3. For $\mathbf{b}$: 5 − 12 + 7 = 0, so it is in the span. Row 2 gives $c_2 = 4 - 1 = 3$, and row 1 gives $c_1 = 1 - 2(3) = -5$.
4. For $\mathbf{d}$: 10 − 3 − 3 = 4 ≠ 0, so it is not in the span.

Answer: the plane $5x - 3y - z = 0$. $\mathbf{b} = -5\mathbf{v}_1 + 3\mathbf{v}_2$ is in it; $\mathbf{d}$ is not. **Check:**
$(-5, -5, -10) + (6, 9, 3) = (1, 4, -7)$, and both $\mathbf{v}_1$ and $\mathbf{v}_2$ satisfy the equation: 5 − 3 − 2 = 0 and 10 − 9 − 1 = 0.

Example 2. Do $\mathbf{w}_1 = (1, 0, 2)$, $\mathbf{w}_2 = (3, 1, 4)$ and $\mathbf{w}_3 = (1, 2, -3)$ span $\mathbb{R}^3$? What if $\mathbf{w}_3$ is replaced by (4, 1, 6)?

1. Put the vectors in as columns. $R_3 - 2R_1$ gives (0, −2, −5), and then $R_3 + 2R_2$ gives (0, 0, −1):
$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 2 & 4 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$
2. There is a pivot in every row, so the three vectors span $\mathbb{R}^3$. No particular $\mathbf{b}$ had to be solved for.
3. With third column (4, 1, 6), the same steps give (0, −2, −2) and then (0, 0, 0). Row 3 has no pivot, so these three do not span $\mathbb{R}^3$.

Answer: the first three vectors span $\mathbb{R}^3$. With (4, 1, 6) they span only a plane. **Check:** $(4, 1, 6) = \mathbf{w}_1 + \mathbf{w}_2$ is already in $\text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$, so by idea 4 it adds nothing.

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