Subspaces

Some sets of vectors are never left by adding and scaling. These subspaces are the flat pieces of space through the origin. How do you prove a set is one, or is not?

WORDS TO KNOW

Subspace of $\mathbb{R}^n$: a set W of vectors in $\mathbb{R}^n$ that contains $\mathbf{0}$ and is closed under addition and scalar multiplication.

Zero subspace: the set $\{\mathbf{0}\}$. It is the smallest subspace.

Closed: the operation never leads out of the set.

Homogeneous system: a system $A\mathbf{x} = \mathbf{0}$, with every right side 0.

KEY IDEAS

1. Three conditions, or one line

W is a subspace of $\mathbb{R}^n$ when

(1) $\mathbf{0}$ is in W; (2) if $\mathbf{u}$ and $\mathbf{v}$ are in W, so is $\mathbf{u} + \mathbf{v}$; (3) if $\mathbf{u}$ is in W, so is $c\mathbf{u}$ for every real c.

One-line test: $\mathbf{0}$ is in W, and $c\mathbf{u} + d\mathbf{v}$ is in W for all $\mathbf{u}, \mathbf{v}$ in W and all scalars c, d.

$\mathbb{R}^n$ itself and $\{\mathbf{0}\}$ are subspaces. To prove a yes, use general vectors, not one example.

2. Spans and homogeneous solution sets are subspaces

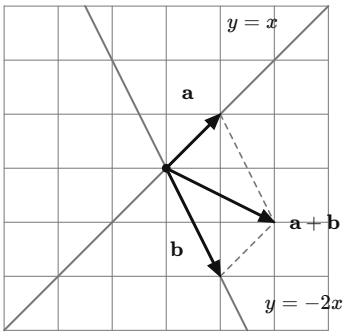
Every span is a subspace. It is the smallest one that holds its vectors.

The solution set of $A\mathbf{x} = \mathbf{0}$ is a subspace. When $\mathbf{b} \neq \mathbf{0}$, the solution set of $A\mathbf{x} = \mathbf{b}$ is not, because $A\mathbf{0} = \mathbf{0} \neq \mathbf{b}$.

To write a subspace given by equations as a span, solve for the leading variables. Each free variable gives one vector.

3. One failing condition, with actual vectors, is enough

A set that misses $\mathbf{0}$ fails at once. A curve such as $y = x^2$, or a set that allows only some scalars, fails scaling. Two different lines through the origin, taken together, fail addition.



4. What the subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$ are

In $\mathbb{R}^2$: $\{\mathbf{0}\}$, the lines through the origin, and $\mathbb{R}^2$.

In $\mathbb{R}^3$: $\{\mathbf{0}\}$, the lines through the origin, the planes through the origin, and $\mathbb{R}^3$.

WATCH OUT

- **Through the origin, always.** A line or plane that misses $\mathbf{0}$ is never a subspace, however flat it is.
- **Every real scalar.** Closure under scaling must hold for negative scalars and for 0 too. A set like the first quadrant, closed only under positive scalars, is not a subspace.
- **A yes needs general vectors.** Checking one or two vectors that work proves nothing. A no needs just one specific failure.
- **The empty set is not a subspace.** It has no $\mathbf{0}$.

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WORKED EXAMPLES

Example 1. Show that the plane P given by $x + 2y - 3z = 0$ is a subspace of $\mathbb{R}^3$, and write it as a span.

1. Zero: 0 + 2(0) − 3(0) = 0, so **0** is in P.
2. Addition: let **u** and **v** be in P. Regroup the terms for $\mathbf{u} + \mathbf{v}$:
$$(u_1 + v_1) + 2(u_2 + v_2) - 3(u_3 + v_3) = (u_1 + 2u_2 - 3u_3) + (v_1 + 2v_2 - 3v_3) = 0 + 0$$
3. Scaling: $cu_1 + 2(cu_2) - 3(cu_3) = c(u_1 + 2u_2 - 3u_3) = c \cdot 0 = 0$.
4. All three hold. For a span, solve for x: $x = -2y + 3z$ with y and z free, so
$$(x, y, z) = (-2y + 3z,\ y,\ z) = y(-2, 1, 0) + z(3, 0, 1)$$

Answer: P is a subspace, and $P = \text{span}\{(-2, 1, 0), (3, 0, 1)\}$. **Check:** −2 + 2 − 0 = 0 and 3 + 0 − 3 = 0.

Example 2. Write the solution set of $A\mathbf{x} = \mathbf{0}$ as a span, where A is the matrix below.
$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 5 & -1 & -1 \end{bmatrix}$$

1. It is a subspace of $\mathbb{R}^4$, since the system is homogeneous. $R_2 - 2R_1$ gives the row (0, 1, 1, −1).
2. Pivots are in columns 1 and 2, so x_3 and x_4 are free. Row 2 gives $x_2 = -x_3 + x_4$.
3. Row 1 gives $x_1 = -2x_2 + x_3 = 3x_3 - 2x_4$.
4. Split by the free variables: $\mathbf{x} = x_3(3, -1, 1, 0) + x_4(-2, 1, 0, 1)$.

Answer: $\text{span}\{(3, -1, 1, 0), (-2, 1, 0, 1)\}$. **Check:** $A(3, -1, 1, 0) = (3 - 2 - 1,\ 6 - 5 - 1) = (0, 0)$ and $A(-2, 1, 0, 1) = (-2 + 2,\ -4 + 5 - 1) = (0, 0)$.

Example 3. Show that none of these is a subspace of $\mathbb{R}^2$: (a) the line $y = 3x - 2$; (b) the half-plane $y \geq 0$; (c) the graph $y = |x|$.

1. (a) 0 ≠ 3(0) − 2, so **0** is not on the line.
2. (b) (3, 2) is in it, but (−1)(3, 2) = (−3, −2) is not, since −2 < 0. Scaling fails.
3. (c) (2, 2) and (−1, 1) are on the graph, but their sum (1, 3) is not, since $|1| \neq 3$. Addition fails.

Answer: each set breaks at least one condition, so none is a subspace.

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