# Adding and scaling vectors in n dimensions

*A day at a creamery gives five numbers, not one. How do you add, compare and scale whole lists of numbers in one move?*

## WORDS TO KNOW

**$\mathbb{R}^n$:** the set of all ordered lists of $n$ real numbers.

**Vector:** one list in $\mathbb{R}^n$, written as a column or as $(u_1, u_2, \ldots, u_n)$.

**Entry:** one number in the list. $u_i$ is the $i$-th entry of $\mathbf{u}$.

**Scalar:** a real number that multiplies a vector.

**Zero vector 0:** the list of $n$ zeros. Each $\mathbb{R}^n$ has its own.

**Negative $-\mathbf{v}$:** the vector $(-1)\mathbf{v}$, with the sign of every entry changed.

## KEY IDEAS

### 1. Equal means equal in every position

A vector is an ordered list, so (1, 2) and (2, 1) are different vectors. Two vectors can be equal only when they have the same number of entries.

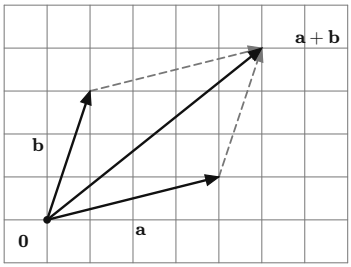
For $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, $\mathbf{x} = \mathbf{y}$ means $x_i = y_i$ for every $i = 1, \ldots, n$.

So one vector equation in $\mathbb{R}^n$ is $n$ number equations, and all of them must hold.

### 2. Add, scale and subtract one position at a time

Add matching entries. Multiply every entry by the scalar. Subtract by adding the negative.

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n)$$
$$c\mathbf{u} = (cu_1, \ldots, cu_n)$$
$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = (u_1 - v_1, \ldots, u_n - v_n)$$




In $\mathbb{R}^2$ adding is the parallelogram rule. Past three entries there is no picture, but the arithmetic is the same.

### 3. Eight properties let you rearrange like numbers

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars $c, d$:

| Adding | Scaling |
|---|---|
| 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | 5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ |
| 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | 6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ |
| 3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ | 7. $c(d\mathbf{u}) = (cd)\mathbf{u}$ |
| 4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ | 8. $1\mathbf{u} = \mathbf{u}$ |

Each one is proved the same way: compare the $i$-th entries of the two sides and use the matching rule for real numbers. The size $n$ never enters, so one proof works in every $\mathbb{R}^n$.

If $c\mathbf{u} = \mathbf{0}$, then $c = 0$ or $\mathbf{u} = \mathbf{0}$.



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## WATCH OUT

- **Only vectors of the same size can be added or compared.** (1, 2, 3) + (1, 2) is not defined. Do not pad the short one with a zero.
- **You cannot add a number to a vector.** $\mathbf{u} + 3$ is not defined. To add 3 to every entry, add (3, 3, ..., 3).
- **Order matters in subtraction.** $\mathbf{u} - \mathbf{v}$ and $\mathbf{v} - \mathbf{u}$ are different vectors unless $\mathbf{u} = \mathbf{v}$.
- **Check every position.** If some positions already fix the unknowns, the other positions are extra conditions. One that fails means there is no solution.

## WORKED EXAMPLES

**Example 1. Let $\mathbf{u} = (3, -2, 0, 5)$ and $\mathbf{v} = (1, 4, -3, -2)$. Find $2\mathbf{u} - 3\mathbf{v}$.**

1. Scale each vector: $2\mathbf{u} = (6, -4, 0, 10)$ and $3\mathbf{v} = (3, 12, -9, -6)$.
2. Subtract position by position: (6 – 3, –4 – 12, 0 – (–9), 10 – (–6)).

**Answer:** $2\mathbf{u} - 3\mathbf{v} = (3, -16, 9, 16)$. **Check:** the last entry is 2(5) – 3(–2) = 10 + 6 = 16.

**Example 2. Find all numbers $x$ and $y$ with $(x + y,\ x - y,\ 3y,\ x + 2y) = (8, 2, 9, 11)$.**

1. Both sides are in $\mathbb{R}^4$, so there are four equations: $x + y = 8$, $x - y = 2$, $3y = 9$ and $x + 2y = 11$.
2. Add the first two: $2x = 10$, so $x = 5$. Then the first gives $y = 3$.
3. The last two are extra conditions. Both hold: 3(3) = 9 and 5 + 2(3) = 11.

**Answer:** $x = 5$ and $y = 3$, the only solution. If the last entry on the right were 12 instead, the first two positions would still force $x = 5$, $y = 3$, and the fourth would fail: no solution.

**Example 3. Solve $4\mathbf{x} + \mathbf{a} = \mathbf{x} + \mathbf{b}$ for $\mathbf{x}$ in $\mathbb{R}^4$, where $\mathbf{a} = (1, -3, 5, 0)$ and $\mathbf{b} = (7, -3, -1, 21)$.**

1. Add $-\mathbf{x}$ and $-\mathbf{a}$ to both sides. Reorder, regroup and cancel (properties 1 to 4), and combine $4\mathbf{x} - \mathbf{x} = 3\mathbf{x}$ (properties 6 and 8): $3\mathbf{x} = \mathbf{b} - \mathbf{a}$.
2. Subtract position by position: $\mathbf{b} - \mathbf{a} = (6, 0, -6, 21)$.
3. Multiply both sides by $\frac{1}{3}$. By properties 7 and 8, $\frac{1}{3}(3\mathbf{x}) = \mathbf{x}$.

**Answer:** $\mathbf{x} = (2, 0, -2, 7)$. **Check:** both sides are (9, –3, –3, 28), since $4\mathbf{x} + \mathbf{a} = (8, 0, -8, 28) + (1, -3, 5, 0)$ and $\mathbf{x} + \mathbf{b} = (2, 0, -2, 7) + (7, -3, -1, 21)$.


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