Linear Algebra Unit 1 quiz 1: Adding and scaling vectors in n dimensions, Dot product, length and orthogonality in n dimensions, Linear combination (answer key)

8 questions, 21 points

ANSWERS AT A GLANCE

1. $(6, -9, 32, 11, -3)$
2. $\cos\theta = -\dfrac{2\sqrt{221}}{51}$; θ is obtuse.
3. No. The first two entries force $c_1 = 2$ and $c_2 = -3$; these fit the third entry but give fourth entry 18, not 16, so no weights work.
4. (a) $k = -3$, $h = 2$. (b) $h = -6$, with $k = -\dfrac{3}{2}$.
5. $\mathbf{x} = \dfrac{1}{3}(\mathbf{v} + 2\mathbf{u}) = (-3, 4, -2, 4)$
6. (a) largest value 9, smallest value 5 (b) for example $\mathbf{p} = (0,0,0,0)$, $\mathbf{q} = (7,0,0,0)$ and $\mathbf{r} = (5,0,0,0)$
7. $\left(\dfrac{1}{5}, \dfrac{2}{5}, -\dfrac{2}{5}, \dfrac{4}{5}\right)$ and $\left(-\dfrac{1}{5}, -\dfrac{2}{5}, \dfrac{2}{5}, -\dfrac{4}{5}\right)$
8. $c_1 = 3$, $c_2 = 3$, $c_3 = 2$: $\mathbf{b} = 3\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3$, the only list of weights.

WORKED SOLUTIONS

1. Let $\mathbf{u} = (2, -3, 6, 1, 1)$ and $\mathbf{v} = (2, -3, -1, -3, 4)$ in $\mathbb{R}^5$. Find $5\mathbf{u} - 2\mathbf{v}$. (2 points)

ANSWER

$(6, -9, 32, 11, -3)$

SOLUTION

Scale each vector: $5\mathbf{u} = (10, -15, 30, 5, 5)$ and $2\mathbf{v} = (4, -6, -2, -6, 8)$.

Subtract position by position: $5\mathbf{u} - 2\mathbf{v} = (6, -9, 32, 11, -3)$.

SCORING

1 point for both scalar multiples; 1 point for the correct difference in all five positions.

2. Let $\mathbf{u} = (-2, -3, 1, -3, 4)$ and $\mathbf{v} = (4, 4, 3, -1, -3)$. Find the exact value of $\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, and say whether θ is acute, right or obtuse. (2 points)

ANSWER

$\cos\theta = -\dfrac{2\sqrt{221}}{51}$; θ is obtuse.

SOLUTION

$\mathbf{u} \cdot \mathbf{v} = -8 - 12 + 3 + 3 - 12 = -26$, $\|\mathbf{u}\|^2 = 39$ and $\|\mathbf{v}\|^2 = 51$. So

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-26}{\sqrt{39 \cdot 51}} = -\frac{2\sqrt{221}}{51}.$$

The dot product is negative, so θ is obtuse.

SCORING

1 point for $\mathbf{u} \cdot \mathbf{v} = -26$ and both lengths; 1 point for $\cos\theta = -\dfrac{2\sqrt{221}}{51}$ in any equal exact form (such as $\dfrac{-26}{\sqrt{39 \cdot 51}}$) with "obtuse".

3. Decide whether $\mathbf{b} = (16, -7, -2, 16)$ is a linear combination of $\mathbf{v}_1 = (2, -2, 2, 3)$ and $\mathbf{v}_2 = (-4, 1, 2, -4)$ in $\mathbb{R}^4$. If it is, find the weights; if it is not, explain why not. (2 points)

ANSWER

No. The first two entries force $c_1 = 2$ and $c_2 = -3$; these fit the third entry but give fourth entry 18, not 16, so no weights work.

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SOLUTION

We need $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{b}$: four equations in the two unknowns c_1, c_2, with augmented matrix

$$\left[\begin{array}{rr|r} 2 & -4 & 16 \\ -2 & 1 & -7 \\ 2 & 2 & -2 \\ 3 & -4 & 16 \end{array}\right]$$

Eliminating in the first two equations allows only $c_1 = 2$ and $c_2 = -3$, so the other two equations must hold for these values or for nothing. The third holds: $2 \cdot 2 + 2 \cdot (-3) = -2$. The fourth fails: $3 \cdot 2 - 4 \cdot (-3) = 18$, not 16.

No weights satisfy all four equations, so $\mathbf{b}$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. (Starting from any other two entries also ends with an entry that fails.)

SCORING

1 point for the weights $c_1 = 2$, $c_2 = -3$ forced by two of the equations (or a reduction of the augmented matrix to an echelon form); 1 point for showing that the fourth entry (or another entry) fails and concluding "not a combination". Checking only the third entry and answering yes earns at most the first point.

4. (3 points)

(a) Find the scalar k and the number h with $k(2, -1, h, 3) + (1, 4, 5, -2) = (-5, 7, -1, -11)$.

(b) Find the number h for which some scalar k satisfies $k(4, -2, 6, 2) = (h, 3, -9, h + 3)$, and find that k.

ANSWER

(a) $k = -3$, $h = 2$. (b) $h = -6$, with $k = -\dfrac{3}{2}$.

SOLUTION

(a) The left side is $(2k + 1, -k + 4, kh + 5, 3k - 2)$. Equality gives four equations: $2k + 1 = -5$, $-k + 4 = 7$, $kh + 5 = -1$ and $3k - 2 = -11$. The first gives $k = -3$, and the second and fourth agree. The third becomes $-3h + 5 = -1$, so $h = 2$.

(b) The positions say $4k = h$, $-2k = 3$, $6k = -9$ and $2k = h + 3$. The second gives $k = -\dfrac{3}{2}$, and the third agrees. The first then gives $h = -6$, and the fourth holds: $2(-\dfrac{3}{2}) = -3 = -6 + 3$. So $h = -6$ and $k = -\dfrac{3}{2}$.

SCORING

1 point for writing one equation per position; 1 point for (a) $k = -3$ and $h = 2$; 1 point for (b) $h = -6$ and $k = -\dfrac{3}{2}$ with every position checked.

5. Solve $2(\mathbf{x} - \mathbf{u}) = \mathbf{v} - \mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^4$, naming the property behind each step, where $\mathbf{u} = (5, -2, 7, 1)$ and $\mathbf{v} = (-19, 16, -20, 10)$. (3 points)

ANSWER

$\mathbf{x} = \dfrac{1}{3}(\mathbf{v} + 2\mathbf{u}) = (-3, 4, -2, 4)$

SOLUTION

Distribute on the left (property 5, with $2(-\mathbf{u}) = -2\mathbf{u}$ by property 7):

$$2\mathbf{x} - 2\mathbf{u} = \mathbf{v} - \mathbf{x}.$$

Add $\mathbf{x} + 2\mathbf{u}$ to both sides. Reorder and regroup each side (properties 1 and 2). On the left, $2\mathbf{x} + \mathbf{x} = (2 +$

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$1)\mathbf{x} = 3\mathbf{x}$ (properties 8 and 6) and $-2\mathbf{u} + 2\mathbf{u} = \mathbf{0}$ (property 4); on the right, $-\mathbf{x} + \mathbf{x} = \mathbf{0}$ (property 4). Dropping the zero vectors (property 3):

$$3\mathbf{x} = \mathbf{v} + 2\mathbf{u}.$$

Multiply both sides by $\frac{1}{3}$. On the left, $\frac{1}{3}(3\mathbf{x}) = (\frac{1}{3} \cdot 3)\mathbf{x}$ by property 7, which is $1\mathbf{x} = \mathbf{x}$ by property 8. So

$$\mathbf{x} = \tfrac{1}{3}(\mathbf{v} + 2\mathbf{u}) = \tfrac{1}{3}(-9, 12, -6, 12) = (-3, 4, -2, 4).$$

Check: $2(\mathbf{x} - \mathbf{u}) = 2(-8, 6, -9, 3) = (-16, 12, -18, 6)$ and $\mathbf{v} - \mathbf{x} = (-16, 12, -18, 6)$.

SCORING

1 point for distributing and collecting the $\mathbf{x}$ terms to reach $3\mathbf{x} = \mathbf{v} + 2\mathbf{u}$, naming properties 5 and 6 (or equivalent); 1 point for scaling by $\frac{1}{3}$ and naming properties 7 and 8; 1 point for $\mathbf{x} = (-3, 4, -2, 4)$.

6. Points $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ of $\mathbb{R}^4$ have $\text{dist}(\mathbf{p}, \mathbf{q}) = 7$ and $\text{dist}(\mathbf{q}, \mathbf{r}) = 2$. (3 points)

(a) Use the triangle inequality to find the largest and the smallest value that $\text{dist}(\mathbf{p}, \mathbf{r})$ can have.

(b) Give points $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ in $\mathbb{R}^4$ with these two distances for which $\text{dist}(\mathbf{p}, \mathbf{r})$ is the smallest value.

ANSWER

(a) largest value 9, smallest value 5 (b) for example $\mathbf{p} = (0, 0, 0, 0)$, $\mathbf{q} = (7, 0, 0, 0)$ and $\mathbf{r} = (5, 0, 0, 0)$

SOLUTION

(a) Theorem 3 in distance form gives $\text{dist}(\mathbf{p}, \mathbf{r}) \leq \text{dist}(\mathbf{p}, \mathbf{q}) + \text{dist}(\mathbf{q}, \mathbf{r}) = 7 + 2 = 9$.

With $\mathbf{r}$ in the middle instead, it gives $\text{dist}(\mathbf{p}, \mathbf{q}) \leq \text{dist}(\mathbf{p}, \mathbf{r}) + \text{dist}(\mathbf{r}, \mathbf{q})$. Since $\mathbf{r} - \mathbf{q} = -(\mathbf{q} - \mathbf{r})$ and $\|c\mathbf{x}\| = |c|\,\|\mathbf{x}\|$, $\text{dist}(\mathbf{r}, \mathbf{q}) = 2$. So $7 \leq \text{dist}(\mathbf{p}, \mathbf{r}) + 2$, that is, $\text{dist}(\mathbf{p}, \mathbf{r}) \geq 5$.

Both bounds are reached by points on one line: with $\mathbf{p} = \mathbf{0}$ and $\mathbf{q} = (7, 0, 0, 0)$, the point (9, 0, 0, 0) is at distance 2 from $\mathbf{q}$ and 9 from $\mathbf{p}$, and the point in (b) gives 5. So the largest value is 9 and the smallest is 5.

(b) Take $\mathbf{p} = (0, 0, 0, 0)$, $\mathbf{q} = (7, 0, 0, 0)$ and $\mathbf{r} = (5, 0, 0, 0)$. Then $\text{dist}(\mathbf{p}, \mathbf{q}) = 7$, $\mathbf{q} - \mathbf{r} = (2, 0, 0, 0)$ has length 2, and $\text{dist}(\mathbf{p}, \mathbf{r}) = 5$. Any example with $\mathbf{r}$ on the segment from $\mathbf{p}$ to $\mathbf{q}$, at distance 2 from $\mathbf{q}$, is correct.

SCORING

1 point for the largest value 9 from the triangle inequality; 1 point for the smallest value 5 from the triangle inequality with $\mathbf{r}$ in the middle (a value with no inequality cited earns nothing); 1 point for an example whose three distances are all checked.

7. Find every unit vector in $\mathbb{R}^4$ that is orthogonal to all three of $\mathbf{a} = (2, -1, 0, 0)$, $\mathbf{b} = (0, 1, 1, 0)$ and $\mathbf{c} = (0, 0, 2, 1)$. (3 points)

ANSWER

$\left(\frac{1}{5}, \frac{2}{5}, -\frac{2}{5}, \frac{4}{5}\right)$ and $\left(-\frac{1}{5}, -\frac{2}{5}, \frac{2}{5}, -\frac{4}{5}\right)$

SOLUTION

A vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$ is orthogonal to all three when its dot product with each is 0:

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ x_2 + x_3 &= 0 \\ 2x_3 + x_4 &= 0 \end{aligned}$$

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The system is already in echelon form with x_1 free. With $x_1 = t$: $x_2 = 2t$, $x_3 = -2t$, $x_4 = 4t$, so $\mathbf{x} = t(1, 2, -2, 4)$.

Its length is $|t|\sqrt{1+4+4+16} = 5|t|$, which is 1 when $t = \dfrac{1}{5}$ or $t = -\dfrac{1}{5}$. The unit vectors are $\left(\dfrac{1}{5}, \dfrac{2}{5}, -\dfrac{2}{5}, \dfrac{4}{5}\right)$ and its negative $\left(-\dfrac{1}{5}, -\dfrac{2}{5}, \dfrac{2}{5}, -\dfrac{4}{5}\right)$.

SCORING

1 point for the three equations from the dot products; 1 point for every solution $t(1, 2, -2, 4)$; 1 point for normalizing to both unit vectors (1 point lost if only one sign is given).

8. Find the weights that write $\mathbf{b} = (6, 1, 9)$ as a linear combination of $\mathbf{v}_1 = (1, -2, 2)$, $\mathbf{v}_2 = (3, 3, 3)$ and $\mathbf{v}_3 = (-3, -1, -3)$. (3 points)

ANSWER

$c_1 = 3$, $c_2 = 3$, $c_3 = 2$: $\mathbf{b} = 3\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3$, the only list of weights.

SOLUTION

The augmented matrix of $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$ has columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{b}$:

$$\left[\begin{array}{ccc|c} 1 & 3 & -3 & 6 \\ -2 & 3 & -1 & 1 \\ 2 & 3 & -3 & 9 \end{array}\right]$$

Clear the first column with $R_2 - (-2)R_1$ and $R_3 - 2R_1$:

$$\left[\begin{array}{ccc|c} 1 & 3 & -3 & 6 \\ 0 & 9 & -7 & 13 \\ 0 & -3 & 3 & -3 \end{array}\right]$$

Eliminating c_2 from the last two rows leaves a pivot in every column, so back substitution gives exactly one solution: $c_3 = 2$, then $c_2 = 3$, then $c_1 = 3$.

Check: 3(1, −2, 2) + 3(3, 3, 3) + 2(−3, −1, −3) = (6, 1, 9).

SCORING

1 point for the system or augmented matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{b}$; 1 point for a correct elimination; 1 point for all three weights 3, 3, 2.

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