Linear Algebra Unit 1 quiz 2: Span, Linear independence (answer key)

7 questions, 20 points

ANSWERS AT A GLANCE

1. (a) For example $(-1, 0) = -\mathbf{v}_1$: its only weights are $c_1 = -1$, $c_2 = 0$. (b) A span that holds $\mathbf{v}_1$ also holds $-\mathbf{v}_1$; W holds $\mathbf{v}_1$ but not $-\mathbf{v}_1$.
2. (a) No: an echelon form of the matrix with these columns has a zero third row, so there is no pivot in row 3. (b) Exactly when $c \neq 0$.
3. (a) $x_1 - 2x_2 + x_3 = 0$ (or any nonzero multiple); $\mathbf{d}$ gives $2 - 2 + 1 = 1 \neq 0$. (b) First entry to 1, or second entry to $\frac{3}{2}$, or third entry to 0; no change of the fourth entry works.
4. They span $\mathbb{R}^2$ exactly when $ab \neq 6$. When $ab = 6$, the span is the line $2x - ay = 0$ through the origin along $(a, 2)$.
5. (a) $c_1 = -3t$, $c_2 = t$, $c_3 = 0$, $c_4 = t$ for $t \neq 0$; for example $3\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_4 = \mathbf{0}$. (b) Leaving out $\mathbf{v}_1$, $\mathbf{v}_2$ or $\mathbf{v}_4$ leaves an independent list; leaving out $\mathbf{v}_3$ leaves the dependent list $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$.
6. (a) Neither is a multiple of the other. (b) $\mathbf{r}_3 = 2\mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{r}_4 = -\mathbf{r}_1 + \mathbf{r}_2$. (c) For example $-\mathbf{r}_1 + 2\mathbf{r}_2 - \mathbf{r}_3 - 3\mathbf{r}_4 = \mathbf{0}$.
7. Only $h = 4$ and $k = 3$. Then $3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$ (or any nonzero multiple).

WORKED SOLUTIONS

1. Let $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (1, 1)$, and let W be the set of all $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ with $c_1 \geq 0$ and $c_2 \geq 0$. (2 points)

(a) Give a vector in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ that is not in W, with a reason.

(b) Explain why W is not the span of any list of vectors.

ANSWER

(a) For example $(-1, 0) = -\mathbf{v}_1$: its only weights are $c_1 = -1$, $c_2 = 0$. (b) A span that holds $\mathbf{v}_1$ also holds $-\mathbf{v}_1$; W holds $\mathbf{v}_1$ but not $-\mathbf{v}_1$.

SOLUTION

(a) $(-1, 0) = -1\mathbf{v}_1 + 0\mathbf{v}_2$ is in the span. To write it as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = (c_1 + c_2, c_2)$, the second entry forces $c_2 = 0$ and then $c_1 = -1$. These are the only weights, and $c_1 < 0$, so (−1, 0) is not in W.

(b) A span is closed under combinations, so if a span contains a vector $\mathbf{w}$ it contains $(-1)\mathbf{w}$. W contains $\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2$ but, by (a), not $-\mathbf{v}_1$. So W is not a span.

SCORING

1 point for a specific vector of the span outside W, with the reason that its only weights include a negative one; 1 point for (b) using closure under the weight −1 with the vector of (a).

2. Let $\mathbf{u}_1 = (1, 2, -1)$, $\mathbf{u}_2 = (2, 3, 1)$, $\mathbf{u}_3 = (0, 1, -3)$ and $\mathbf{u}_4 = (3, 5, 0)$. (3 points)

(a) Do these four vectors span $\mathbb{R}^3$? Use an echelon form to decide.

(b) Keep $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$, and replace $\mathbf{u}_4$ by $(3, 5, c)$. For which numbers c do the four vectors span $\mathbb{R}^3$?

ANSWER

(a) No: an echelon form of the matrix with these columns has a zero third row, so there is no pivot in row 3. (b) Exactly when $c \neq 0$.

SOLUTION

(a) Put the vectors in as columns. $R_2 - 2R_1$ and $R_3 + R_1$, then $R_3 + 3R_2$:

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 3 & 1 & 5 \\ -1 & 1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -1 & 1 & -1 \\ 0 & 3 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row 3 has no pivot, so the vectors do not span $\mathbb{R}^3$, even though there are four of them.

(b) The same row operations act on the new fourth column $(3, 5, c)$: row 2 becomes 5 − 6 = −1, row 3 becomes $c + 3$ and then $c + 3 + 3(-1) = c$. So row 3 of the echelon form is $(0, 0, 0, c)$. It has a pivot

Free sample from math.wiki for classroom use. Do not post. math.wiki

exactly when $c \neq 0$, so the four vectors span $\mathbb{R}^3$ exactly when $c \neq 0$.

SCORING

1 point for a correct echelon form in (a); 1 point for "no" with the reason that row 3 has no pivot ("no" from counting vectors alone earns nothing); 1 point for $c \neq 0$ with the reason that the last entry of row 3 becomes c.

3. Let $\mathbf{v}_1 = (1, 0, -1, 2)$, $\mathbf{v}_2 = (0, 1, 2, -1)$ and $\mathbf{v}_3 = (1, 1, 1, 0)$ in $\mathbb{R}^4$, and let $\mathbf{d} = (2, 1, 1, 0)$. (3 points)

(a) Find one equation in x_1, x_2, x_3, x_4 satisfied by exactly the vectors of $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and use it to show that $\mathbf{d}$ is not in the span.

(b) Changing exactly one entry of $\mathbf{d}$ can put it in the span. For each of the four entries, say what it must be changed to, or that no change of that entry works.

ANSWER

(a) $x_1 - 2x_2 + x_3 = 0$ (or any nonzero multiple); $\mathbf{d}$ gives 2 − 2 + 1 = 1 ≠ 0. (b) First entry to 1, or second entry to $\dfrac{3}{2}$, or third entry to 0; no change of the fourth entry works.

SOLUTION

(a) Reduce $[\,\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3 \mid \mathbf{x}\,]$ with a general $\mathbf{x} = (x_1, x_2, x_3, x_4)$. First $R_3 + R_1$ and $R_4 - 2R_1$; then $R_3 - 2R_2$ and $R_4 + R_2$, and swap rows 3 and 4:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & x_1 \\ 0 & 1 & 1 & x_2 \\ -1 & 2 & 1 & x_3 \\ 2 & -1 & 0 & x_4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & x_1 \\ 0 & 1 & 1 & x_2 \\ 0 & 2 & 2 & x_3 + x_1 \\ 0 & -1 & -2 & x_4 - 2x_1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & x_1 \\ 0 & 1 & 1 & x_2 \\ 0 & 0 & -1 & x_4 - 2x_1 + x_2 \\ 0 & 0 & 0 & x_1 - 2x_2 + x_3 \end{array}\right]$$

The system is consistent exactly when the last entry is 0, so the span is $x_1 - 2x_2 + x_3 = 0$. Check: $\mathbf{v}_1$ gives 1 − 0 − 1 = 0, $\mathbf{v}_2$ gives 0 − 2 + 2 = 0, $\mathbf{v}_3$ gives 1 − 2 + 1 = 0. For $\mathbf{d}$, 2 − 2 + 1 = 1 ≠ 0, so $\mathbf{d}$ is not in the span.

(b) Keep three entries of $\mathbf{d}$ and solve the equation for the fourth. First entry t: $t - 2 + 1 = 0$, so $t = 1$. Second entry t: $2 - 2t + 1 = 0$, so $t = \dfrac{3}{2}$. Third entry t: $2 - 2 + t = 0$, so $t = 0$. The fourth entry does not appear in the equation, so changing it leaves the value $1 \neq 0$: no change of it works.

SCORING

1 point for a correct reduction with a general right side giving $x_1 - 2x_2 + x_3 = 0$ (or a multiple); 1 point for $\mathbf{d}$ not in the span with the value $1 \neq 0$ (or the row 0 = 1); 1 point for all of (b): the three new entries 1, $\dfrac{3}{2}$, 0 and that the fourth entry cannot be fixed.

4. For which real numbers a and b do $(a, 2)$ and $(3, b)$ span $\mathbb{R}^2$? For every other pair a, b, describe their span by an equation. Use row reduction, not a formula. (3 points)

ANSWER

They span $\mathbb{R}^2$ exactly when $ab \neq 6$. When $ab = 6$, the span is the line $2x - ay = 0$ through the origin along $(a, 2)$.

SOLUTION

Put the vectors in as columns of $\begin{bmatrix} a & 3 \\ 2 & b \end{bmatrix}$.

If $a = 0$, swap the rows: $\begin{bmatrix} 2 & b \\ 0 & 3 \end{bmatrix}$ has a pivot in every row, so the vectors span $\mathbb{R}^2$ (and $ab = 0 \neq 6$).

Free sample from math.wiki for classroom use. Do not post. math.wiki

If $a \neq 0$, replace row 2 by $R_2 - \dfrac{2}{a}R_1$:

$$\begin{bmatrix} a & 3 \\ 2 & b \end{bmatrix} \sim \begin{bmatrix} a & 3 \\ 0 & b - \frac{6}{a} \end{bmatrix}$$

Row 2 has a pivot exactly when $b - \dfrac{6}{a} \neq 0$, that is, $ab \neq 6$. So in every case the vectors span $\mathbb{R}^2$ exactly when $ab \neq 6$.

When $ab = 6$, $a \neq 0$ and $b = \dfrac{6}{a}$, so $(3, b) = \dfrac{3}{a}(a, 2)$ and the span is the line through the origin along $(a, 2)$. With a right side (x, y) the same step leaves $[\,0 \;\; 0 \mid y - \dfrac{2}{a}x\,]$, so the line is $y - \dfrac{2}{a}x = 0$, or $2x - ay = 0$.

SCORING

1 point for a correct reduction that handles $a = 0$ separately (or never divides by a); 1 point for "span $\mathbb{R}^2$ exactly when $ab \neq 6$"; 1 point for the line $2x - ay = 0$ (or an equivalent equation) when $ab = 6$.

5. Let $\mathbf{v}_1 = (1, -2, 0, 3)$, $\mathbf{v}_2 = (2, -3, 1, 4)$, $\mathbf{v}_3 = (0, 1, 2, -1)$ and $\mathbf{v}_4 = (1, -3, -1, 5)$ in $\mathbb{R}^4$. (3 points)

(a) Find every dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

(b) For each of the four vectors, decide whether the list of the OTHER three is linearly independent.

ANSWER

(a) $c_1 = -3t$, $c_2 = t$, $c_3 = 0$, $c_4 = t$ for $t \neq 0$; for example $3\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_4 = \mathbf{0}$. (b) Leaving out $\mathbf{v}_1$, $\mathbf{v}_2$ or $\mathbf{v}_4$ leaves an independent list; leaving out $\mathbf{v}_3$ leaves the dependent list $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$.

SOLUTION

(a) Row reduce the matrix with these columns: $R_2 + 2R_1$ and $R_4 - 3R_1$, then $R_3 - R_2$ and $R_4 + 2R_2$, then $R_4 - R_3$:

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ -2 & -3 & 1 & -3 \\ 0 & 1 & 2 & -1 \\ 3 & 4 & -1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & -2 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Column 4 has no pivot, so $c_4 = t$ is free. Row 3: $c_3 = 0$. Row 2: $c_2 + c_3 - c_4 = 0$, so $c_2 = t$. Row 1: $c_1 + 2c_2 + c_4 = 0$, so $c_1 = -3t$. The relations are $-3t\,\mathbf{v}_1 + t\,\mathbf{v}_2 + 0\,\mathbf{v}_3 + t\,\mathbf{v}_4 = \mathbf{0}$ with $t \neq 0$; $t = -1$ gives $3\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_4 = \mathbf{0}$. Check: (3, −6, 0, 9) − (2, −3, 1, 4) − (1, −3, −1, 5) = (0, 0, 0, 0).

(b) A relation among three of the vectors is a relation among all four with weight 0 on the one left out. Leaving out $\mathbf{v}_1$ needs $-3t = 0$, so $t = 0$ and every weight is 0: $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ is independent. The same holds for leaving out $\mathbf{v}_2$ ($t = 0$) or $\mathbf{v}_4$ ($t = 0$). Leaving out $\mathbf{v}_3$ costs nothing, since $c_3 = 0$ in every relation, so $3\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_4 = \mathbf{0}$ shows $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$ is dependent.

SCORING

1 point for a correct echelon form with pivots in columns 1, 2 and 3; 1 point for every relation (a multiple of (−3, 1, 0, 1), or the one relation with "and its nonzero multiples"); 1 point for (b): independent when $\mathbf{v}_1$, $\mathbf{v}_2$ or $\mathbf{v}_4$ is left out, dependent when $\mathbf{v}_3$ is, with a reason.

6. A chemist records a reaction as a vector: for the substances in the order $(\mathrm{C}, \mathrm{O_2}, \mathrm{CO}, \mathrm{CO_2})$, each entry is the number of moles produced (positive) or used up (negative). Four reactions of carbon and oxygen give - $\mathbf{r}_1 = (-1, -1, 0, 1)$ for $\mathrm{C} + \mathrm{O_2} \to \mathrm{CO_2}$, - $\mathbf{r}_2 = (-2, -1, 2, 0)$ for $2\mathrm{C} + \mathrm{O_2} \to 2\mathrm{CO}$, - $\mathbf{r}_3 = (0, -1, -2, 2)$ for $2\mathrm{CO} + \mathrm{O_2} \to 2\mathrm{CO_2}$, - $\mathbf{r}_4 = (-1, 0, 2, -1)$ for $\mathrm{C} + \mathrm{CO_2} \to 2\mathrm{CO}$. (3 points)

(a) Show that $\mathbf{r}_1, \mathbf{r}_2$ is linearly independent.

Free sample from math.wiki for classroom use. Do not post. math.wiki

(b) Write $\mathbf{r}_3$ and $\mathbf{r}_4$ each as a linear combination of $\mathbf{r}_1$ and $\mathbf{r}_2$.

(c) Find a dependence relation among $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4$ in which all four weights are nonzero.

ANSWER

(a) Neither is a multiple of the other. (b) $\mathbf{r}_3 = 2\mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{r}_4 = -\mathbf{r}_1 + \mathbf{r}_2$. (c) For example $-\mathbf{r}_1 + 2\mathbf{r}_2 - \mathbf{r}_3 - 3\mathbf{r}_4 = \mathbf{0}$.

SOLUTION

(a) Two vectors are dependent exactly when one is a multiple of the other (Result 2). The third entries are 0 and 2, so $\mathbf{r}_2$ is not a multiple of $\mathbf{r}_1$; the fourth entries are 1 and 0, so $\mathbf{r}_1$ is not a multiple of $\mathbf{r}_2$. Independent.

(b) For $\mathbf{r}_3 = c_1\mathbf{r}_1 + c_2\mathbf{r}_2$, the fourth entry gives $c_1 = 2$ and the third gives $2c_2 = -2$, so $c_2 = -1$. Check all entries: 2(−1, −1, 0, 1) − (−2, −1, 2, 0) = (0, −1, −2, 2). For $\mathbf{r}_4$, the fourth entry gives $c_1 = -1$ and the third $c_2 = 1$; check: −(−1, −1, 0, 1) + (−2, −1, 2, 0) = (−1, 0, 2, −1).

(c) (b) gives two relations, $2\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{r}_3 = \mathbf{0}$ and $-\mathbf{r}_1 + \mathbf{r}_2 - \mathbf{r}_4 = \mathbf{0}$. Their sum $\mathbf{r}_1 - \mathbf{r}_3 - \mathbf{r}_4 = \mathbf{0}$ has weight 0 on $\mathbf{r}_2$, so take the first plus 3 times the second:

$$-\mathbf{r}_1 + 2\mathbf{r}_2 - \mathbf{r}_3 - 3\mathbf{r}_4 = \mathbf{0}.$$

Check: (1, 1, 0, −1) + (−4, −2, 4, 0) + (0, 1, 2, −2) + (3, 0, −6, 3) = (0, 0, 0, 0). In chemical terms, reaction 3 adds nothing new: it is two runs of reaction 1 followed by reaction 2 run backward.

SCORING

1 point for (a) with the reason (neither vector is a multiple of the other, or the only weights are 0); 1 point for both combinations in (b); 1 point for a correct relation in (c) with all four weights nonzero (any one that sums to $\mathbf{0}$ is accepted).

7. Let $\mathbf{v}_1 = (1, 2, -1, 0)$, $\mathbf{v}_2 = (-1, -1, 3, 2)$ and $\mathbf{v}_3 = (1, h, k, 4)$ in $\mathbb{R}^4$. Find all pairs of numbers h and k for which the list $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly dependent, and for those values give a dependence relation. (3 points)

ANSWER

Only $h = 4$ and $k = 3$. Then $3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$ (or any nonzero multiple).

SOLUTION

Row reduce the matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$: $R_2 - 2R_1$ and $R_3 + R_1$, then $R_3 - 2R_2$ and $R_4 - 2R_2$:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & h \\ -1 & 3 & k \\ 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & h-2 \\ 0 & 2 & k+1 \\ 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & h-2 \\ 0 & 0 & k-2h+5 \\ 0 & 0 & 8-2h \end{bmatrix}$$

Columns 1 and 2 always have pivots. Column 3 has no pivot only when both $8 - 2h$ and $k - 2h + 5$ are zero: $h = 4$, and then $k = 2h - 5 = 3$. One condition is not enough: with $h = 4$ but $k \neq 3$, the entry $k - 3$ is a pivot.

Then $c_3 = t$ is free, row 2 gives $c_2 + 2t = 0$, so $c_2 = -2t$, and row 1 gives $c_1 = c_2 - c_3 = -3t$. With $t = -1$: $3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$. Check: (3, 6, −3, 0) + (−2, −2, 6, 4) − (1, 4, 3, 4) = (0, 0, 0, 0).

SCORING

1 point for a correct echelon form with $k - 2h + 5$ and $8 - 2h$ (or equivalent expressions) below the pivots of column 3; 1 point for "dependent exactly when $h = 4$ and $k = 3$" (giving only $h = 4$, or a condition linking h and k alone, earns no point here); 1 point for a correct dependence relation.

Free sample from math.wiki for classroom use. Do not post. math.wiki