Linear Algebra Unit 1 quiz 3: Subspaces, Basis and dimension (answer key)

7 questions, 20 points

ANSWERS AT A GLANCE

1. (a) No: $(1,2,1) \in S$ but $2(1,2,1) = (2,4,2) \notin S$. (b) Yes: $T = \text{span}\{(1,2,1)\}$, a line through the origin.
2. (a) $N = \text{span}\{(-3,-1,1,0), (1,2,0,1)\}$ (any list with the same span is accepted). (b) $h = -7$, $k = -4$.
3. (a) The condition is equivalent to $\mathbf{x} \cdot \mathbf{a} = 0$, that is, $x - 2y + 2z = 0$. (b) $W = \text{span}\{(2,1,0), (-2,0,1)\}$ (any list with the same span is accepted).
4. (a) Both satisfy the equation, neither is a multiple of the other, and $\dim P = 2$, so they are a basis by Result 6. (b) $(3, -2)$. (c) $(1, 0, 0)$ is not in P.
5. (a) $(-1, 1, 1, 0)$, $(3, -2, 0, 1)$, so $\dim V = 2$ (any basis of V is accepted). (b) For example add $\mathbf{e}_1$ and $\mathbf{e}_2$: the four vectors are independent, and four independent vectors in $\mathbb{R}^4$ are a basis (any two vectors that make the four independent are accepted).
6. $\dim N_k = 2$ for $k \neq 4$, and $\dim N_4 = 3$. A basis of N_4: $(-2, 1, 0, 0)$, $(1, 0, 1, 0)$, $(-1, 0, 0, 1)$ (any basis of N_4 is accepted).
7. See the proof below.

WORKED SOLUTIONS

1. Decide whether each set is a subspace of $\mathbb{R}^3$. Give a reason for a yes and specific failing vectors for a no. (2 points)

(a) S, the set of all vectors $(t, 2t, t^3)$ for real t.

(b) T, the set of all vectors $(t^3, 2t^3, t^3)$ for real t.

> **ANSWER**
> (a) No: $(1,2,1) \in S$ but $2(1,2,1) = (2,4,2) \notin S$. (b) Yes: $T = \text{span}\{(1,2,1)\}$, a line through the origin.
>
> **SOLUTION**
> (a) $t = 1$ gives $(1,2,1) \in S$. If $(2,4,2) = (t, 2t, t^3)$, the first entry forces $t = 2$, and then the third entry would be 8, not 2. So $2(1,2,1) = (2,4,2) \notin S$: S is not closed under scaling.
>
> (b) $(t^3, 2t^3, t^3) = t^3(1,2,1)$, and t^3 takes every real value as t runs over the reals (each real c is t^3 for $t = \sqrt[3]{c}$). So T is exactly the set of all $c(1,2,1)$, which is $\text{span}\{(1,2,1)\}$. Every span is a subspace.
>
> **SCORING**
> 1 point for (a) "no" with a specific vector of S whose multiple (or a sum of two vectors) is shown not to be in S; 1 point for (b) "yes" with the reason that t^3 takes every real value, so T is a span. A bare yes or no earns nothing.

2. Let

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & 5 & 0 \\ 1 & 0 & 3 & -1 \end{bmatrix}.$$

(a) Write the solution set N of $A\mathbf{x} = \mathbf{0}$ as a span.

(b) Find the numbers h and k for which $(h, k, 2, -1)$ is in N. (3 points)

> **ANSWER**
> (a) $N = \text{span}\{(-3,-1,1,0), (1,2,0,1)\}$ (any list with the same span is accepted). (b) $h = -7$, $k = -4$.
>
> **SOLUTION**
> (a) Row reduce with $R_2 - 2R_1$ and $R_3 - R_1$, then $R_3 - R_2$:
>
> $$A \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

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The third equation was a combination of the first two. With x_3 and x_4 free, row 2 gives $x_2 = -x_3 + 2x_4$, and row 1 gives $x_1 = x_2 - 2x_3 - x_4 = -3x_3 + x_4$. So

$$\mathbf{x} = x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix},$$

and $N = \text{span}\{(-3,-1,1,0),(1,2,0,1)\}$. Check: $A(-3,-1,1,0) = (-3+1+2, -6+1+5, -3+3) = (0,0,0)$ and $A(1,2,0,1) = (1-2+1, 2-2, 1-1) = (0,0,0)$.

(b) A vector of N with $x_3 = 2$ and $x_4 = -1$ has $x_1 = -6 - 1 = -7$ and $x_2 = -2 - 2 = -4$: it is 2(−3, −1, 1, 0) − (1, 2, 0, 1) = (−7, −4, 2, −1). So $h = -7$ and $k = -4$. Check the first row: −7 + 4 + 4 − 1 = 0.

SCORING

1 point for a correct reduction to an echelon form with two free variables; 1 point for two spanning vectors of N (any correct pair with the same span); 1 point for $h = -7$, $k = -4$.

3. Let $\mathbf{a} = (1,-2,2)$, and let W be the set of vectors $\mathbf{x}$ in $\mathbb{R}^3$ with $\|\mathbf{x}+\mathbf{a}\| = \|\mathbf{x}-\mathbf{a}\|$. (3 points)

(a) Show that W is the solution set of one homogeneous linear equation in x, y, z, and so is a subspace of $\mathbb{R}^3$.

(b) Write W as a span.

ANSWER

(a) The condition is equivalent to $\mathbf{x} \cdot \mathbf{a} = 0$, that is, $x - 2y + 2z = 0$. (b) $W = \text{span}\{(2,1,0),(-2,0,1)\}$ (any list with the same span is accepted).

SOLUTION

(a) Lengths are never negative, so the two lengths are equal exactly when their squares are. Expanding with the dot product,

$$\|\mathbf{x}+\mathbf{a}\|^2 = \|\mathbf{x}\|^2 + 2\,\mathbf{x}\cdot\mathbf{a} + \|\mathbf{a}\|^2,$$
$$\|\mathbf{x}-\mathbf{a}\|^2 = \|\mathbf{x}\|^2 - 2\,\mathbf{x}\cdot\mathbf{a} + \|\mathbf{a}\|^2.$$

They are equal exactly when $4\,\mathbf{x}\cdot\mathbf{a} = 0$, that is, $\mathbf{x}\cdot\mathbf{a} = 0$: with $\mathbf{x} = (x,y,z)$, the equation $x - 2y + 2z = 0$. The solution set of a homogeneous equation is a subspace.

(b) Solve for x: $x = 2y - 2z$, with y and z free. Every vector of W is

$$(2y - 2z,\ y,\ z) = y(2,1,0) + z(-2,0,1),$$

so $W = \text{span}\{(2,1,0),(-2,0,1)\}$. Check: 2 − 2 + 0 = 0 and −2 − 0 + 2 = 0.

SCORING

1 point for squaring and expanding to $\mathbf{x}\cdot\mathbf{a} = 0$; 1 point for the equation $x - 2y + 2z = 0$ with the reason it gives a subspace (homogeneous equation, or the three conditions); 1 point for a correct span.

4. Let P be the plane $3x + y - 2z = 0$ in $\mathbb{R}^3$, and let $\mathbf{b}_1 = (1,-1,1)$ and $\mathbf{b}_2 = (2,0,3)$. (3 points)

(a) Show that $\mathbf{b}_1, \mathbf{b}_2$ is a basis of P. Check that they lie in P and are independent, and find $\dim P$; do not check spanning directly.

(b) Find the coordinates of (−1, −3, −3) relative to $\mathbf{b}_1, \mathbf{b}_2$.

(c) Explain why (1, 0, 0) has no coordinates relative to $\mathbf{b}_1, \mathbf{b}_2$.

ANSWER

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(a) Both satisfy the equation, neither is a multiple of the other, and dim $P = 2$, so they are a basis by Result 6. (b) (3, −2). (c) (1, 0, 0) is not in P.

SOLUTION

(a) 3 − 1 − 2 = 0 and 6 + 0 − 6 = 0, so both lie in P. The second entries are −1 and 0, so $\mathbf{b}_1$ is not a multiple of $\mathbf{b}_2$, and the first entries 1 and 2 with third entries 1 and 3 show $\mathbf{b}_2$ is not a multiple of $\mathbf{b}_1$: independent. Solving $y = -3x + 2z$ with x and z free, $P = \text{span}\{(1, -3, 0), (0, 2, 1)\}$, two independent vectors by Result 9(b), so dim $P = 2$. Two independent vectors in a subspace of dimension 2 span it (Result 6), so $\mathbf{b}_1, \mathbf{b}_2$ is a basis of P.

(b) $c_1(1, -1, 1) + c_2(2, 0, 3) = (-1, -3, -3)$. The second entry gives $-c_1 = -3$, so $c_1 = 3$; the first gives $3 + 2c_2 = -1$, so $c_2 = -2$. The third entries check: 3 − 6 = −3. The coordinates are (3, −2).

(c) 3(1) + 0 − 0 = 3 ≠ 0, so (1, 0, 0) is not in P. Every combination of $\mathbf{b}_1$ and $\mathbf{b}_2$ lies in the subspace P, so no weights give (1, 0, 0). Coordinates exist only for vectors of the subspace the basis belongs to.

SCORING

1 point for (a): membership, independence and dim $P = 2$ with Result 6 (all three needed); 1 point for the coordinates (3, −2); 1 point for (c): (1, 0, 0) fails the equation, so it is not a combination of $\mathbf{b}_1, \mathbf{b}_2$.

5. Let V be the set of vectors $\mathbf{x}$ in $\mathbb{R}^4$ with $x_1 + x_2 - x_4 = 0$ and $x_2 - x_3 + 2x_4 = 0$. (3 points)

(a) Find a basis of V and give dim V.

(b) Extend your basis to a basis of $\mathbb{R}^4$, and explain why the longer list is a basis.

ANSWER

(a) (−1, 1, 1, 0), (3, −2, 0, 1), so dim $V = 2$ (any basis of V is accepted). (b) For example add $\mathbf{e}_1$ and $\mathbf{e}_2$: the four vectors are independent, and four independent vectors in $\mathbb{R}^4$ are a basis (any two vectors that make the four independent are accepted).

SOLUTION

(a) The pivots are at x_1 and x_2; x_3 and x_4 are free. The second equation gives $x_2 = x_3 - 2x_4$, and the first gives $x_1 = -x_2 + x_4 = -x_3 + 3x_4$. So

$$\mathbf{x} = x_3(-1, 1, 1, 0) + x_4(3, -2, 0, 1),$$

and by Result 9(b) these two vectors are a basis of V: dim $V = 2$.

(b) Take (−1, 1, 1, 0), (3, −2, 0, 1), $\mathbf{e}_1$, $\mathbf{e}_2$. In a zero combination $a(-1, 1, 1, 0) + b(3, -2, 0, 1) + c\,\mathbf{e}_1 + d\,\mathbf{e}_2 = \mathbf{0}$, the third entry gives $a = 0$ and the fourth gives $b = 0$; then the first two entries give $c = 0$ and $d = 0$. So the four vectors are independent, and four independent vectors in $\mathbb{R}^4$, where dim $\mathbb{R}^4 = 4$, span it by Result 6. They are a basis of $\mathbb{R}^4$.

SCORING

1 point for a correct basis of V; 1 point for dim $V = 2$; 1 point for two added vectors with a reason the four are a basis (independence plus the count, Result 6). Adding vectors with no reason earns nothing for (b).

6. For each real number k, let N_k be the set of solutions in $\mathbb{R}^4$ of

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 0 \\ 2x_1 + kx_2 - 2x_3 + 2x_4 &= 0. \end{aligned}$$

Find dim N_k for every value of k, and give a basis of N_k for the value of k where the dimension is largest. (3 points)

ANSWER

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dim $N_k = 2$ for $k \neq 4$, and dim $N_4 = 3$. A basis of N_4: (−2, 1, 0, 0), (1, 0, 1, 0), (−1, 0, 0, 1) (any basis of N_4 is accepted).

SOLUTION

Subtract twice the first equation from the second: $(k-4)x_2 = 0$.

If $k \neq 4$, the echelon form has pivots at x_1 and x_2, so x_3 and x_4 are free and dim $N_k = 4 - 2 = 2$ by Result 9(b).

If $k = 4$, the second equation becomes 0 = 0 and only the first is left, with one pivot at x_1; x_2, x_3, x_4 are free and dim $N_4 = 3$. From $x_1 = -2x_2 + x_3 - x_4$,

$$\mathbf{x} = x_2(-2,1,0,0) + x_3(1,0,1,0) + x_4(-1,0,0,1),$$

so these three vectors are a basis of N_4.

SCORING

1 point for dim $N_k = 2$ when $k \neq 4$ with the pivot count; 1 point for dim $N_4 = 3$; 1 point for a correct basis of N_4 (three independent solutions).

7. Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{v}$ be a vector of $\mathbb{R}^n$ that is not in W. Let $\mathbf{v} + W$ be the set of all vectors $\mathbf{v} + \mathbf{w}$ with $\mathbf{w} \in W$. Prove that $\mathbf{v} + W$ is not a subspace. (3 points)

ANSWER

$\mathbf{0} \notin \mathbf{v} + W$: if $\mathbf{0} = \mathbf{v} + \mathbf{w}$ with $\mathbf{w} \in W$, then $\mathbf{v} = (-1)\mathbf{w} \in W$, a contradiction.

SOLUTION

It is enough to show that $\mathbf{0}$ is not in $\mathbf{v} + W$. Suppose it were: $\mathbf{0} = \mathbf{v} + \mathbf{w}$ for some $\mathbf{w} \in W$. Adding $-\mathbf{w}$ to both sides gives $\mathbf{v} = -\mathbf{w} = (-1)\mathbf{w}$. W is a subspace, so it is closed under scaling by –1, and $(-1)\mathbf{w} \in W$. That puts $\mathbf{v}$ in W, which contradicts the given fact that $\mathbf{v} \notin W$.

So $\mathbf{0} \notin \mathbf{v} + W$, and $\mathbf{v} + W$ fails the first condition: it is not a subspace.

SCORING

1 point for aiming at the zero vector (writing $\mathbf{0} = \mathbf{v} + \mathbf{w}$ with $\mathbf{w} \in W$); 1 point for $\mathbf{v} = (-1)\mathbf{w} \in W$ by closure under scaling (the scaling must be named); 1 point for the contradiction with $\mathbf{v} \notin W$ and the conclusion. An example with particular vectors instead of a proof earns at most 1 point.

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