Linear Algebra Unit 1 test: Vectors in n dimensions (answer key)

12 questions, 52 points

ANSWERS AT A GLANCE

1. $\mathbf{p} = (2, -6, 4, 0, 10)$ and $\mathbf{q} = (10, 2, -4, 8, -6)$
2. Largest value 15, at $\left(\frac{6}{5}, -\frac{3}{5}, \frac{6}{5}, \frac{12}{5}\right)$; smallest value −15, at $\left(-\frac{6}{5}, \frac{3}{5}, -\frac{6}{5}, -\frac{12}{5}\right)$.
3. Diego: $4\mathbf{a} + 3\mathbf{b} + \mathbf{c}$, so 4 of A, 3 of B and 1 of C. Jenny: only $-\mathbf{a} + 3\mathbf{b} + 3\mathbf{c}$, which needs −1 pack, so no. Samir: only $\frac{10}{3}\mathbf{a} + \frac{10}{3}\mathbf{b} + \frac{2}{3}\mathbf{c}$, which needs parts of packs, so no.
4. (a) $h \neq -4$, any k: $c_1 = \frac{3h + 2k}{h + 4}$, $c_2 = \frac{k - 6}{h + 4}$. (b) $h = -4$ and $k = 6$: $c_1 = 3 + 2t$, $c_2 = t$ for any real t. (c) $h = -4$ and $k \neq 6$.
5. (a) $\mathbf{w}_1 = \mathbf{u}_1 + \mathbf{u}_2$ and $\mathbf{w}_2 = 2\mathbf{u}_1 - \mathbf{u}_2$; $\mathbf{u}_1 = \frac{1}{3}\mathbf{w}_1 + \frac{1}{3}\mathbf{w}_2$ and $\mathbf{u}_2 = \frac{2}{3}\mathbf{w}_1 - \frac{1}{3}\mathbf{w}_2$; so each span contains the other. (b) No: $\text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ is the line along (1, 2, 5), and $\mathbf{u}_1$ is not on it.
6. (a) $\mathbf{a} \cdot \mathbf{b} = 0$, $\mathbf{a} \cdot \mathbf{c} = 0$, $\mathbf{b} \cdot \mathbf{c} = 0$. (b) Expand by the properties of Theorem 1; the six cross terms vanish. (c) $\sqrt{93}$
7. (a) $x_1 + x_2 - x_3 = 0$ and $x_1 - x_2 - x_4 = 0$ (or any equivalent pair). (b) $h = 2$, $k = -1$: $(1, 2, 3, -1) = -5\mathbf{v}_1 + 3\mathbf{v}_2$ (other weights also work, for example $-3\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3$). (c) No: the row operations of (a) leave rows 3 and 4 zero in the first three columns, so with any fourth column an echelon form has no pivot in row 4.
8. (a) (−1, 1, 0, 0), (1, 0, 1, 0), (−2, 0, 0, 1), so $\dim H = 3$ (any basis of H is accepted). (b) Every combination of $\mathbf{a}_1, \mathbf{a}_2$ has fourth entry 0. If $w_4 \neq 0$, $\mathbf{w}$ is outside their span, so the three are independent and, with $\dim H = 3$, a basis by Result 6. If $w_4 = 0$, $\mathbf{w}$ lies in the subspace K of vectors of H with $x_4 = 0$, which has dimension 2 and so equals $\text{span}\{\mathbf{a}_1, \mathbf{a}_2\}$ by Result 7; then the three are dependent. (c) (−1, 1, 1).
9. $h = 3$ and $h = -1$. For $h = 3$: $\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$. For $h = -1$: $-3\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$. (Any nonzero multiples.)
10. Only $k = 2$. Then W_2 is the plane $x - 2z = 0$, and $W_2 = \text{span}\{(0, 1, 0), (2, 0, 1)\}$ (any list with the same span is accepted).
11. See the proof below.
12. See the proof below.

WORKED SOLUTIONS

1. Two points $\mathbf{p}$ and $\mathbf{q}$ of $\mathbb{R}^5$ are unknown. The midpoint of the segment from $\mathbf{p}$ to $\mathbf{q}$ is $\mathbf{m} = (6, -2, 0, 4, 2)$, and the point one quarter of the way from $\mathbf{p}$ to $\mathbf{q}$ is $\mathbf{r} = (4, -4, 2, 2, 6)$. Find $\mathbf{p}$ and $\mathbf{q}$. (4 points)

ANSWER

$\mathbf{p} = (2, -6, 4, 0, 10)$ and $\mathbf{q} = (10, 2, -4, 8, -6)$

SOLUTION

The two facts are vector equations:

$$\mathbf{m} = \tfrac{1}{2}(\mathbf{p} + \mathbf{q}), \qquad \mathbf{r} = \mathbf{p} + \tfrac{1}{4}(\mathbf{q} - \mathbf{p}) = \tfrac{3}{4}\mathbf{p} + \tfrac{1}{4}\mathbf{q}.$$

The first gives $\mathbf{q} = 2\mathbf{m} - \mathbf{p}$. Substitute into the second:

$$\mathbf{r} = \tfrac{3}{4}\mathbf{p} + \tfrac{1}{2}\mathbf{m} - \tfrac{1}{4}\mathbf{p} = \tfrac{1}{2}\mathbf{p} + \tfrac{1}{2}\mathbf{m},$$

so $\mathbf{r}$ is the midpoint of $\mathbf{p}$ and $\mathbf{m}$, and $\mathbf{p} = 2\mathbf{r} - \mathbf{m}$:

$$\mathbf{p} = (8, -8, 4, 4, 12) - (6, -2, 0, 4, 2) = (2, -6, 4, 0, 10).$$

Then $\mathbf{q} = 2\mathbf{m} - \mathbf{p} = (12, -4, 0, 8, 4) - (2, -6, 4, 0, 10) = (10, 2, -4, 8, -6)$.

Check: $\frac{1}{2}(\mathbf{p} + \mathbf{q}) = \frac{1}{2}(12, -4, 0, 8, 4) = \mathbf{m}$, and $\mathbf{p} + \frac{1}{4}(\mathbf{q} - \mathbf{p}) = (2, -6, 4, 0, 10) + (2, 2, -2, 2, -4) = \mathbf{r}$.

SCORING

1 point for $\mathbf{m} = \frac{1}{2}(\mathbf{p} + \mathbf{q})$ or an equivalent equation; 1 point for $\mathbf{r} = \frac{3}{4}\mathbf{p} + \frac{1}{4}\mathbf{q}$ or an equivalent equation; 1 point for $\mathbf{p} = (2, -6, 4, 0, 10)$; 1 point for $\mathbf{q} = (10, 2, -4, 8, -6)$. Solving position by position, five times, also earns full credit.

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2. Over all real numbers x_1, x_2, x_3, x_4 with $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 9$, find the largest and the smallest value of $2x_1 - x_2 + 2x_3 + 4x_4$, and the numbers $x_1, \ldots, x_4$ where each is reached. Justify with the Cauchy-Schwarz inequality. (4 points)

ANSWER

Largest value 15, at $\left(\frac{6}{5}, -\frac{3}{5}, \frac{6}{5}, \frac{12}{5}\right)$; smallest value −15, at $\left(-\frac{6}{5}, \frac{3}{5}, -\frac{6}{5}, -\frac{12}{5}\right)$.

SOLUTION

The expression is a dot product: with $\mathbf{c} = (2, -1, 2, 4)$ and $\mathbf{x} = (x_1, x_2, x_3, x_4)$, it is $\mathbf{c} \cdot \mathbf{x}$. Here $\|\mathbf{c}\| = \sqrt{4 + 1 + 4 + 16} = 5$ and $\|\mathbf{x}\| = \sqrt{9} = 3$.

By the Cauchy-Schwarz inequality, $|\mathbf{c} \cdot \mathbf{x}| \le 5 \cdot 3 = 15$, so $-15 \le \mathbf{c} \cdot \mathbf{x} \le 15$.

Equality needs $\mathbf{x}$ to be a multiple of $\mathbf{c}$, say $\mathbf{x} = s\mathbf{c}$. Then $\|\mathbf{x}\| = 5|s| = 3$, so $s = \pm\frac{3}{5}$, and $\mathbf{c} \cdot \mathbf{x} = s\|\mathbf{c}\|^2 = 25s = \pm 15$. So both bounds are reached:

$s = \frac{3}{5}$ gives $\mathbf{x} = \left(\frac{6}{5}, -\frac{3}{5}, \frac{6}{5}, \frac{12}{5}\right)$ and the value 15, the largest;

$s = -\frac{3}{5}$ gives $\mathbf{x} = \left(-\frac{6}{5}, \frac{3}{5}, -\frac{6}{5}, -\frac{12}{5}\right)$ and the value −15, the smallest.

Check: $\frac{36 + 9 + 36 + 144}{25} = 9$, and $\frac{12 + 3 + 12 + 48}{5} = 15$.

SCORING

1 point for writing the expression as $\mathbf{c} \cdot \mathbf{x}$ with $\|\mathbf{c}\| = 5$ and $\|\mathbf{x}\| = 3$; 1 point for the bound $|\mathbf{c} \cdot \mathbf{x}| \le 15$ from the Cauchy-Schwarz inequality; 1 point for using the equality case, $\mathbf{x} = \pm\frac{3}{5}\mathbf{c}$; 1 point for both values with their points. Values found by trial with no inequality earn at most the last point.

3. A hardware supplier sells three packs. In numbers of (bolts, nuts, washers), pack A holds $\mathbf{a} = (1, 0, 2)$, pack B holds $\mathbf{b} = (0, 1, 1)$ and pack C holds $\mathbf{c} = (1, 1, 0)$. Packs are sold whole and cannot be opened or returned. Diego needs exactly (5, 4, 11), Jenny exactly (2, 6, 1) and Samir exactly (4, 4, 10). For each order, find every list of weights that writes it as a linear combination of **a**, **b** and **c**, and decide whether it can be filled with packs. (4 points)

ANSWER

Diego: $4\mathbf{a} + 3\mathbf{b} + \mathbf{c}$, so 4 of A, 3 of B and 1 of C. Jenny: only $-\mathbf{a} + 3\mathbf{b} + 3\mathbf{c}$, which needs −1 pack, so no. Samir: only $\frac{10}{3}\mathbf{a} + \frac{10}{3}\mathbf{b} + \frac{2}{3}\mathbf{c}$, which needs parts of packs, so no.

SOLUTION

For an order (x, y, z), $c_1\mathbf{a} + c_2\mathbf{b} + c_3\mathbf{c} = (x, y, z)$ means

$$c_1 + c_3 = x, \qquad c_2 + c_3 = y, \qquad 2c_1 + c_2 = z.$$

The first two give $c_1 = x - c_3$ and $c_2 = y - c_3$; the third becomes $2x + y - 3c_3 = z$. So every order has exactly one list of weights:

$$c_3 = \frac{2x + y - z}{3}, \qquad c_1 = x - c_3, \qquad c_2 = y - c_3.$$

An order can be filled exactly when all three are whole numbers ≥ 0.

Diego, (5, 4, 11): $c_3 = \frac{10 + 4 - 11}{3} = 1$, $c_1 = 4$, $c_2 = 3$. Check: (4, 0, 8) + (0, 3, 3) + (1, 1, 0) = (5, 4, 11). He buys 4 A, 3 B, 1 C.

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Jenny, (2, 6, 1): $c_3 = \dfrac{4+6-1}{3} = 3$, $c_1 = -1$, $c_2 = 3$. Check: (−1, 0, −2) + (0, 3, 3) + (3, 3, 0) = (2, 6, 1). The only list needs −1 pack of A, so it cannot be filled.

Samir, (4, 4, 10): $c_3 = \dfrac{8+4-10}{3} = \dfrac{2}{3}$, $c_1 = c_2 = \dfrac{10}{3}$. The only list uses fractions of packs, so it cannot be filled.

SCORING

1 point for the system and its solution showing each order has exactly one list of weights; 1 point for Diego's packs 4, 3, 1; 1 point for Jenny's weights with "no" because of the negative weight; 1 point for Samir's weights with "no" because they are not whole numbers.

4. Let $\mathbf{v}_1 = (1, -1, 2)$, $\mathbf{v}_2 = (-2, 2, h)$ and $\mathbf{b} = (3, -3, k)$. Find all values of h and k for which $\mathbf{b}$ is (4 points)

(a) a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ with exactly one list of weights (give the weights in terms of h and k);

(b) a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ with more than one list of weights;

(c) not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

ANSWER

(a) $h \neq -4$, any k: $c_1 = \dfrac{3h+2k}{h+4}$, $c_2 = \dfrac{k-6}{h+4}$. (b) $h = -4$ and $k = 6$: $c_1 = 3 + 2t$, $c_2 = t$ for any real t. (c) $h = -4$ and $k \neq 6$.

SOLUTION

The augmented matrix of $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{b}$, reduced with $R_2 + R_1$ and $R_3 - 2R_1$:

$$\left[\begin{array}{rr|r} 1 & -2 & 3 \\ -1 & 2 & -3 \\ 2 & h & k \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & h+4 & k-6 \end{array}\right]$$

So the system is $c_1 - 2c_2 = 3$ and $(h+4)c_2 = k - 6$.

(a) If $h \neq -4$, the second equation forces $c_2 = \dfrac{k-6}{h+4}$, and then $c_1 = 3 + 2c_2 = \dfrac{3h+12+2k-12}{h+4} = \dfrac{3h+2k}{h+4}$: exactly one list, for every k.

(b) If $h = -4$ and $k = 6$, the second equation reads 0 = 0, so $c_2 = t$ is free and $c_1 = 3 + 2t$. For example $t = 0$ gives $\mathbf{b} = 3\mathbf{v}_1$.

(c) If $h = -4$ and $k \neq 6$, the second equation reads $0 = k - 6 \neq 0$, so no weights exist.

SCORING

1 point for a correct reduction to $c_1 - 2c_2 = 3$ and $(h+4)c_2 = k - 6$; 1 point for case (a) with both weights; 1 point for case (b) with its family of weights; 1 point for case (c).

5. Let $\mathbf{u}_1 = (1, 0, 1)$, $\mathbf{u}_2 = (0, 1, 2)$, $\mathbf{w}_1 = (1, 1, 3)$, $\mathbf{w}_2 = (2, -1, 0)$, $\mathbf{x}_1 = (1, 2, 5)$ and $\mathbf{x}_2 = (-2, -4, -10)$. (4 points)

(a) Prove that $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$.

(b) Is $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$? Explain.

ANSWER

(a) $\mathbf{w}_1 = \mathbf{u}_1 + \mathbf{u}_2$ and $\mathbf{w}_2 = 2\mathbf{u}_1 - \mathbf{u}_2$; $\mathbf{u}_1 = \dfrac{1}{3}\mathbf{w}_1 + \dfrac{1}{3}\mathbf{w}_2$ and $\mathbf{u}_2 = \dfrac{2}{3}\mathbf{w}_1 - \dfrac{1}{3}\mathbf{w}_2$; so each span contains the other. (b) No: $\text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ is the line along (1, 2, 5), and $\mathbf{u}_1$ is not on it.

SOLUTION

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(a) Each $\mathbf{w}$ is a combination of the $\mathbf{u}$'s: $\mathbf{u}_1 + \mathbf{u}_2 = (1,1,3) = \mathbf{w}_1$ and $2\mathbf{u}_1 - \mathbf{u}_2 = (2,-1,0) = \mathbf{w}_2$. (Equivalently, reducing $[\,\mathbf{u}_1\ \mathbf{u}_2 \mid (x,y,z)\,]$ gives the plane $x + 2y - z = 0$, and both $\mathbf{w}$'s satisfy it.)

Each $\mathbf{u}$ is a combination of the $\mathbf{w}$'s. For $\mathbf{u}_1$, $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 = (c_1 + 2c_2, c_1 - c_2, 3c_1) = (1,0,1)$ gives $c_1 = \dfrac{1}{3}$ and $c_2 = \dfrac{1}{3}$, and the first entry checks: $\dfrac{1}{3} + \dfrac{2}{3} = 1$. For $\mathbf{u}_2 = (0,1,2)$, the third entry gives $c_1 = \dfrac{2}{3}$, the second $c_2 = -\dfrac{1}{3}$, and the first checks: $\dfrac{2}{3} - \dfrac{2}{3} = 0$.

Now apply the adding result twice. Since $\mathbf{w}_1, \mathbf{w}_2$ are in $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$,

$$\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{w}_1, \mathbf{w}_2\} = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\},$$

and since $\mathbf{u}_1, \mathbf{u}_2$ are in $\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$, the same four-vector span equals $\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$. So the two spans are equal.

(b) No. $\mathbf{x}_2 = -2\mathbf{x}_1$, so $\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2\} = \operatorname{span}\{\mathbf{x}_1\}$, the line of multiples $(c, 2c, 5c)$. $\mathbf{u}_1 = (1,0,1)$ is not on it: the first entry forces $c = 1$, and then the second entry would be 2, not 0. So $\mathbf{u}_1$ is in the first span but not the second. (The line does lie inside the plane, since 1 + 4 − 5 = 0, but it is not all of it.)

SCORING

1 point for showing $\mathbf{w}_1, \mathbf{w}_2$ are in $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ (weights or the plane equation); 1 point for showing $\mathbf{u}_1, \mathbf{u}_2$ are in $\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$ with weights; 1 point for concluding equality from both inclusions (the adding result or closure under combinations). Only one inclusion, or "both are planes", earns no point for the conclusion. 1 point for (b): "no" with a specific vector, such as $\mathbf{u}_1$, in one span and not the other.

6. Let $\mathbf{a} = (1,0,2,0,1)$, $\mathbf{b} = (2,1,-1,0,0)$ and $\mathbf{c} = (1,-2,0,1,-1)$ in $\mathbb{R}^5$. (5 points)

(a) Show that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are pairwise orthogonal.

(b) Prove that if $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in $\mathbb{R}^n$ are pairwise orthogonal, then for all scalars p, q, r,

$$\|p\mathbf{x} + q\mathbf{y} + r\mathbf{z}\|^2 = p^2\|\mathbf{x}\|^2 + q^2\|\mathbf{y}\|^2 + r^2\|\mathbf{z}\|^2.$$

(c) Use (b) to find $\|2\mathbf{a} - \mathbf{b} + 3\mathbf{c}\|$.

ANSWER

(a) $\mathbf{a}\cdot\mathbf{b} = 0$, $\mathbf{a}\cdot\mathbf{c} = 0$, $\mathbf{b}\cdot\mathbf{c} = 0$. (b) Expand by the properties of Theorem 1; the six cross terms vanish. (c) $\sqrt{93}$

SOLUTION

(a) $\mathbf{a}\cdot\mathbf{b} = 2 + 0 - 2 + 0 + 0 = 0$, $\mathbf{a}\cdot\mathbf{c} = 1 + 0 + 0 + 0 - 1 = 0$ and $\mathbf{b}\cdot\mathbf{c} = 2 - 2 + 0 + 0 + 0 = 0$.

(b) Let $\mathbf{w} = p\mathbf{x} + q\mathbf{y} + r\mathbf{z}$. Then $\|\mathbf{w}\|^2 = \mathbf{w}\cdot\mathbf{w}$. By the distributive property, used on the first factor and (with symmetry) on the second, $\mathbf{w}\cdot\mathbf{w}$ is the sum of the nine products $(p\mathbf{x})\cdot(p\mathbf{x})$, $(p\mathbf{x})\cdot(q\mathbf{y})$, and so on. By property 3 each of these is a product of scalars times a dot product, for example $(p\mathbf{x})\cdot(q\mathbf{y}) = pq\,(\mathbf{x}\cdot\mathbf{y})$. The six cross terms contain $\mathbf{x}\cdot\mathbf{y}$, $\mathbf{y}\cdot\mathbf{x}$, $\mathbf{x}\cdot\mathbf{z}$, $\mathbf{z}\cdot\mathbf{x}$, $\mathbf{y}\cdot\mathbf{z}$ or $\mathbf{z}\cdot\mathbf{y}$, each 0 by orthogonality (and symmetry). What remains is

$$\mathbf{w}\cdot\mathbf{w} = p^2\,\mathbf{x}\cdot\mathbf{x} + q^2\,\mathbf{y}\cdot\mathbf{y} + r^2\,\mathbf{z}\cdot\mathbf{z} = p^2\|\mathbf{x}\|^2 + q^2\|\mathbf{y}\|^2 + r^2\|\mathbf{z}\|^2.$$

(c) $\|\mathbf{a}\|^2 = 1 + 4 + 1 = 6$, $\|\mathbf{b}\|^2 = 4 + 1 + 1 = 6$ and $\|\mathbf{c}\|^2 = 1 + 4 + 1 + 1 = 7$. By (b) with $p = 2$, $q = -1$, $r = 3$,

$$\|2\mathbf{a} - \mathbf{b} + 3\mathbf{c}\|^2 = 4\cdot 6 + 1\cdot 6 + 9\cdot 7 = 93,$$

so the length is $\sqrt{93}$. Check from the entries: $2\mathbf{a} - \mathbf{b} + 3\mathbf{c} = (3,-7,5,3,-1)$, and 9 + 49 + 25 + 9 + 1 = 93.

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SCORING
(a) 1 point for all three dot products equal to 0. (b) 1 point for expanding $\mathbf{w} \cdot \mathbf{w}$ into nine terms by the distributive property; 1 point for bringing out the scalars by property 3 and removing the six cross terms by orthogonality; 1 point for the conclusion with $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$. A check on the vectors of (a) alone is not a proof and earns no point in (b). (c) 1 point for $\sqrt{93}$ with the squared lengths 6, 6, 7.

7. Let $\mathbf{v}_1 = (1, -1, 0, 2)$, $\mathbf{v}_2 = (2, -1, 1, 3)$ and $\mathbf{v}_3 = (0, 1, 1, -1)$ in $\mathbb{R}^4$. (5 points)

(a) Find two equations in x_1, x_2, x_3, x_4 satisfied by exactly the vectors of $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) Find the numbers h and k for which $(1, h, 3, k)$ is in the span, and write that vector as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(c) Is there a vector $\mathbf{w}$ in $\mathbb{R}^4$ for which $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}$ span $\mathbb{R}^4$? Explain.

ANSWER
(a) $x_1 + x_2 - x_3 = 0$ and $x_1 - x_2 - x_4 = 0$ (or any equivalent pair). (b) $h = 2$, $k = -1$: $(1, 2, 3, -1) = -5\mathbf{v}_1 + 3\mathbf{v}_2$ (other weights also work, for example $-3\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3$). (c) No: the row operations of (a) leave rows 3 and 4 zero in the first three columns, so with any fourth column an echelon form has no pivot in row 4.

SOLUTION
(a) Reduce with a general right side $\mathbf{x} = (x_1, x_2, x_3, x_4)$. First $R_2 + R_1$ and $R_4 - 2R_1$, then $R_3 - R_2$ and $R_4 + R_2$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & x_1 \\ -1 & -1 & 1 & x_2 \\ 0 & 1 & 1 & x_3 \\ 2 & 3 & -1 & x_4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & x_1 \\ 0 & 1 & 1 & x_1 + x_2 \\ 0 & 1 & 1 & x_3 \\ 0 & -1 & -1 & x_4 - 2x_1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & x_1 \\ 0 & 1 & 1 & x_1 + x_2 \\ 0 & 0 & 0 & x_3 - x_1 - x_2 \\ 0 & 0 & 0 & x_4 - x_1 + x_2 \end{array}\right]$$

The system is consistent exactly when both zero rows have right side 0:

$$x_1 + x_2 - x_3 = 0, \qquad x_1 - x_2 - x_4 = 0.$$

Check: $\mathbf{v}_1$ gives 0 and 1 + 1 − 2 = 0; $\mathbf{v}_2$ gives 2 − 1 − 1 = 0 and 2 + 1 − 3 = 0; $\mathbf{v}_3$ gives 0 + 1 − 1 = 0 and 0 − 1 + 1 = 0.

(b) $1 + h - 3 = 0$ gives $h = 2$, and $1 - 2 - k = 0$ gives $k = -1$. For (1, 2, 3, −1), row 2 says $c_2 + c_3 = 3$ and row 1 says $c_1 + 2c_2 = 1$, with c_3 free. Taking $c_3 = 0$: $c_2 = 3$ and $c_1 = -5$, so $(1, 2, 3, -1) = -5\mathbf{v}_1 + 3\mathbf{v}_2$. Check: (−5, 5, 0, −10) + (6, −3, 3, 9) = (1, 2, 3, −1).

(c) No. Apply the row operations of (a) to $[\,\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{w}\,]$ for any $\mathbf{w}$. Rows 3 and 4 become zero in the first three columns, so only their fourth entries can be nonzero. If both are nonzero, subtracting a multiple of row 3 from row 4 clears row 4. Either way an echelon form has a zero row 4, so there is no pivot in row 4 and the four vectors do not span $\mathbb{R}^4$, whatever $\mathbf{w}$ is.

(Equally: $\mathbf{v}_3 = \mathbf{v}_2 - 2\mathbf{v}_1$, since (2, −1, 1, 3) − (2, −2, 0, 4) = (0, 1, 1, −1). So for any $\mathbf{w}$, $\mathbf{v}_3$ is in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}\}$, and by the adding result the four vectors span the same set as the three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}$. Three vectors never span $\mathbb{R}^4$.)

SCORING
1 point for a correct reduction carrying a general right side; 1 point for both equations (any equivalent pair); 1 point for $h = 2$, $k = -1$ with one correct list of weights; 1 point for "no" in (c); 1 point for a reason that works for every $\mathbf{w}$ (the pivot count with a fourth column, or the adding result with $\mathbf{v}_3 = \mathbf{v}_2 - 2\mathbf{v}_1$ checked by entries and "three vectors never span $\mathbb{R}^4$"). "No, four vectors are needed" or a single trial $\mathbf{w}$ earns no reason point.

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8. Let H be the set of vectors in $\mathbb{R}^4$ with $x_1 + x_2 - x_3 + 2x_4 = 0$, and let $\mathbf{a}_1 = (1, 0, 1, 0)$ and $\mathbf{a}_2 = (2, 1, 3, 0)$. (5 points)

(a) Find a basis of H and give $\dim H$.

(b) Prove that a vector $\mathbf{w}$ of H can be added to $\mathbf{a}_1, \mathbf{a}_2$ to give a basis of H exactly when the fourth entry of $\mathbf{w}$ is not 0.

(c) With $\mathbf{w} = (0, 0, 2, 1)$, find the coordinates of (1, 1, 4, 1) relative to the basis $\mathbf{a}_1, \mathbf{a}_2, \mathbf{w}$ of H.

ANSWER

(a) (−1, 1, 0, 0), (1, 0, 1, 0), (−2, 0, 0, 1), so $\dim H = 3$ (any basis of H is accepted). (b) Every combination of $\mathbf{a}_1, \mathbf{a}_2$ has fourth entry 0. If $w_4 \neq 0$, $\mathbf{w}$ is outside their span, so the three are independent and, with $\dim H = 3$, a basis by Result 6. If $w_4 = 0$, $\mathbf{w}$ lies in the subspace K of vectors of H with $x_4 = 0$, which has dimension 2 and so equals $\text{span}\{\mathbf{a}_1, \mathbf{a}_2\}$ by Result 7; then the three are dependent. (c) (−1, 1, 1).

SOLUTION

(a) Solve for x_1: $x_1 = -x_2 + x_3 - 2x_4$, with x_2, x_3, x_4 free. So

$$\mathbf{x} = x_2(-1, 1, 0, 0) + x_3(1, 0, 1, 0) + x_4(-2, 0, 0, 1),$$

and by Result 9(b) these three vectors are a basis of H: $\dim H = 3$.

(b) Both $\mathbf{a}_i$ lie in H: 1 + 0 − 1 + 0 = 0 and 2 + 1 − 3 + 0 = 0. They are independent: a multiple of $\mathbf{a}_1$ has second entry 0, but $\mathbf{a}_2$ has second entry 1, and a multiple of $\mathbf{a}_2$ equal to $\mathbf{a}_1$ would need factor 0 (second entries) and then 1 = 0. Every combination is

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = (c_1 + 2c_2,\ c_2,\ c_1 + 3c_2,\ 0),$$

with fourth entry 0.

If $w_4 \neq 0$. Then $\mathbf{w}$ is not in $\text{span}\{\mathbf{a}_1, \mathbf{a}_2\}$, so by the lemma for growing a list, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{w}$ is independent. Three independent vectors in H, which has dimension 3 by (a), are a basis of H by Result 6.

If $w_4 = 0$. Let K be the set of vectors with $x_1 + x_2 - x_3 + 2x_4 = 0$ and $x_4 = 0$: the solution set of a homogeneous system, so a subspace. Its vectors are $(x_1, x_2, x_1 + x_2, 0) = x_1(1, 0, 1, 0) + x_2(0, 1, 1, 0)$, and these two vectors are independent (the first two entries read x_1 and x_2), so $\dim K = 2$. Both $\mathbf{a}_i$ lie in K, so $\text{span}\{\mathbf{a}_1, \mathbf{a}_2\} \subseteq K$, and both have dimension 2; by Result 7 they are equal. So $\mathbf{w}$, which lies in K, is a combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, and $\mathbf{a}_1, \mathbf{a}_2, \mathbf{w}$ is dependent: not a basis.

(c) $\mathbf{w} = (0, 0, 2, 1)$ is in H (0 + 0 − 2 + 2 = 0) and has fourth entry 1, so by (b) the list is a basis of H. Then

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{w} = (c_1 + 2c_2,\ c_2,\ c_1 + 3c_2 + 2c_3,\ c_3).$$

Matching (1, 1, 4, 1): $c_3 = 1$, $c_2 = 1$, $c_1 = 1 - 2 = -1$, and the third entry checks: −1 + 3 + 2 = 4. The coordinates are (−1, 1, 1).

SCORING

1 point for (a): three independent solutions of the equation and $\dim H = 3$; 1 point for the "if" direction of (b): $\mathbf{w}$ is outside the span because every combination of $\mathbf{a}_1, \mathbf{a}_2$ ends in 0, then the lemma and Result 6 with $\dim H = 3$; 1 point for showing that the vectors of H with fourth entry 0 are exactly the combinations of $\mathbf{a}_1, \mathbf{a}_2$ (a dimension count with Result 7, or solving for the weights $c_2 = x_2$, $c_1 = x_1 - 2x_2$ directly); 1 point for concluding that such a $\mathbf{w}$ makes the list dependent, so not a basis; 1 point for the coordinates (−1, 1, 1). A direction asserted without a reason earns nothing.

9. Let $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 2, h)$ and $\mathbf{v}_3 = (1, h, 5)$. Find every number h for which $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly dependent, and for each such h give a dependence relation with integer weights. (4 points)

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ANSWER
$h = 3$ and $h = -1$. For $h = 3$: $\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$. For $h = -1$: $-3\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$. (Any nonzero multiples.)

SOLUTION
Row reduce the matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$: $R_2 - R_1$ and $R_3 - R_1$, then $R_3 - (h-1)R_2$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & h \\ 1 & h & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & h-1 \\ 0 & h-1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & h-1 \\ 0 & 0 & 4-(h-1)^2 \end{bmatrix}$$

Columns 1 and 2 always have pivots. Column 3 has none exactly when $4 - (h-1)^2 = 0$, that is $(h-1)^2 = 4$, so $h = 3$ or $h = -1$. For every other h the list is independent.

With $c_3 = t$ free, row 2 gives $c_2 = -(h-1)t$ and row 1 gives $c_1 = -c_2 - c_3 = (h-2)t$.

$h = 3$: $c_2 = -2t$, $c_1 = t$. With $t = 1$, $\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = (1,1,1) - (2,4,6) + (1,3,5) = (0,0,0)$.

$h = -1$: $c_2 = 2t$, $c_1 = -3t$. With $t = 1$, $-3\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = (-3,-3,-3) + (2,4,-2) + (1,-1,5) = (0,0,0)$.

SCORING
1 point for a correct reduction ending in $4 - (h-1)^2$ (or an equivalent such as $(3-h)(h+1)$) in the last pivot place; 1 point for both values $h = 3$ and $h = -1$ (one value alone earns no point); 1 point for a correct relation when $h = 3$; 1 point for a correct relation when $h = -1$.

10. For each real number k, let W_k be the set of vectors (x, y, z) in $\mathbb{R}^3$ with

$$x - 2z + (k-2)y^2 = k^2 - 4.$$

Find every value of k for which W_k is a subspace of $\mathbb{R}^3$. Prove that your values work and that every other value fails, and write W_k as a span for each value that works. (5 points)

ANSWER
Only $k = 2$. Then W_2 is the plane $x - 2z = 0$, and $W_2 = \text{span}\{(0,1,0), (2,0,1)\}$ (any list with the same span is accepted).

SOLUTION
The zero vector. $(0, 0, 0)$ gives 0 on the left, so $\mathbf{0} \in W_k$ only when $k^2 - 4 = 0$, that is, $k = 2$ or $k = -2$. Every other k fails.

$k = -2$. The condition is $x - 2z - 4y^2 = 0$. The vector $(4, 1, 0)$ is in W_{-2}: 4 − 0 − 4 = 0. But $2(4, 1, 0) = (8, 2, 0)$ gives 8 − 0 − 16 = −8 ≠ 0, so it is not. W_{-2} is not closed under scaling, and is not a subspace.

$k = 2$. The condition is $x - 2z = 0$, a homogeneous linear equation, so W_2 is a subspace (its solution set). Solving $x = 2z$ with y and z free,

$$(2z, y, z) = y(0,1,0) + z(2,0,1),$$

so $W_2 = \text{span}\{(0,1,0), (2,0,1)\}$. Check: 0 − 0 = 0 and 2 − 2 = 0.

So W_k is a subspace for $k = 2$ only.

SCORING
1 point for using $\mathbf{0}$ to rule out every k other than ± 2; 1 point for showing W_{-2} fails with specific vectors (a vector of W_{-2} whose multiple, or a pair whose sum, is not in it; "it contains y^2" alone earns nothing); 1

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point for why W_2 is a subspace (a homogeneous linear equation, or the three conditions); 1 point for a correct span of W_2; 1 point for the conclusion "only $k = 2$".

11. Let $\mathbf{u}$ be a nonzero vector in $\mathbb{R}^n$. (4 points)

(a) Find every scalar c with $c^2\mathbf{u} = (c+6)\mathbf{u}$. Justify each step from the eight properties and the fact that $c\mathbf{u} = \mathbf{0}$ only when $c = 0$ or $\mathbf{u} = \mathbf{0}$.

(b) Which scalars c satisfy $c^2\mathbf{u} = (c+6)\mathbf{u}$ when $\mathbf{u}$ is instead the zero vector? Explain.

ANSWER

(a) $c = 3$ or $c = -2$. (b) Every real number c, since both sides are $\mathbf{0}$.

SOLUTION

(a) Add $-((c+6)\mathbf{u})$, which is $(-(c+6))\mathbf{u}$ by property 7, to both sides. On the right, property 4 gives $\mathbf{0}$. On the left, property 6 gives

$$c^2\mathbf{u} + (-(c+6))\mathbf{u} = (c^2 - c - 6)\mathbf{u}.$$

So $(c^2 - c - 6)\mathbf{u} = \mathbf{0}$. Since $\mathbf{u} \neq \mathbf{0}$, the scalar must be zero: $c^2 - c - 6 = (c-3)(c+2) = 0$, so $c = 3$ or $c = -2$.

Both work: $c = 3$ gives $9\mathbf{u} = 9\mathbf{u}$, and $c = -2$ gives $4\mathbf{u} = 4\mathbf{u}$.

(b) For any scalar d, $d\mathbf{0} = \mathbf{0}$, since each entry is $d \cdot 0 = 0$. So with $\mathbf{u} = \mathbf{0}$ both sides are $\mathbf{0}$ whatever c is, and every real c satisfies the equation. The step in (a) that used $\mathbf{u} \neq \mathbf{0}$ is exactly the one that fails.

SCORING

1 point for reaching $(c^2 - c - 6)\mathbf{u} = \mathbf{0}$ with the properties named; 1 point for using $\mathbf{u} \neq \mathbf{0}$ to conclude $c^2 - c - 6 = 0$; 1 point for $c = 3$ and $c = -2$; 1 point for (b) every c, with the reason that both sides are $\mathbf{0}$. Answers with no reasons earn at most the third point.

12. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be a linearly independent list in $\mathbb{R}^n$, let c be a number, and put

$$\mathbf{w} = 2\mathbf{v}_1 - \mathbf{v}_2 + c\,\mathbf{v}_3.$$

(a) Find every value of c for which the list $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}$ is linearly independent, and prove your answer. For any other value, give a dependence relation.

(b) Prove that the list $\mathbf{v}_1, \mathbf{w}, \mathbf{v}_3$ is linearly independent for every c. (4 points)

ANSWER

(a) Independent exactly when $c \neq 0$. For $c = 0$: $2\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{w} = \mathbf{0}$. (b) In any relation $x\mathbf{v}_1 + y\mathbf{w} + z\mathbf{v}_3 = \mathbf{0}$ the weight on $\mathbf{v}_2$ after substituting is $-y$, so $y = 0$, and then $x = z = 0$.

SOLUTION

(a) Suppose $x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{w} = \mathbf{0}$. Substituting for $\mathbf{w}$ and collecting the weights on $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$,

$$(x + 2z)\mathbf{v}_1 + (y - z)\mathbf{v}_2 + cz\,\mathbf{v}_3 = \mathbf{0}.$$

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is independent, each weight is 0:

$$x + 2z = 0, \qquad y - z = 0, \qquad cz = 0.$$

If $c \neq 0$, the last equation gives $z = 0$, and then $x = -2z = 0$ and $y = z = 0$. Only the trivial combination gives $\mathbf{0}$, so $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}$ is independent.

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If $c = 0$, then $\mathbf{w} = 2\mathbf{v}_1 - \mathbf{v}_2$, so

$$2\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{w} = \mathbf{0},$$

a relation with weight $-1 \neq 0$ on $\mathbf{w}$ (it is $x = 2$, $y = -1$, $z = -1$ in the equations above). So the list is independent exactly when $c \neq 0$.

(b) Suppose $x\mathbf{v}_1 + y\mathbf{w} + z\mathbf{v}_3 = \mathbf{0}$. Substituting for $\mathbf{w}$,

$$(x + 2y)\mathbf{v}_1 - y\,\mathbf{v}_2 + (cy + z)\mathbf{v}_3 = \mathbf{0},$$

and independence of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ gives $x + 2y = 0$, $-y = 0$ and $cy + z = 0$. So $y = 0$, then $x = 0$ and $z = 0$, whatever c is: the list is independent. The difference from (a) is that $\mathbf{w}$ is now the only vector in the list that carries $\mathbf{v}_2$, with the nonzero weight -1.

SCORING

1 point for rewriting the relation in (a) in terms of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ with correct weights and using their independence to set each weight to 0 (this step must be stated, not assumed); 1 point for "independent exactly when $c \neq 0$", with $z = 0$ and then $x = y = 0$ shown; 1 point for the explicit relation $2\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{w} = \mathbf{0}$ when $c = 0$; 1 point for (b) by the same method, valid for every c. Testing only particular vectors such as $\mathbf{v}_i = \mathbf{e}_i$ earns at most the third point.

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