Linear Algebra Unit 1 test: Vectors in n dimensions

Name _____ Date _____ Period _____

12 questions, 52 points

1. Two points $\mathbf{p}$ and $\mathbf{q}$ of $\mathbb{R}^5$ are unknown. The midpoint of the segment from $\mathbf{p}$ to $\mathbf{q}$ is $\mathbf{m} = (6, -2, 0, 4, 2)$, and the point one quarter of the way from $\mathbf{p}$ to $\mathbf{q}$ is $\mathbf{r} = (4, -4, 2, 2, 6)$. Find $\mathbf{p}$ and $\mathbf{q}$. (4 points)

2. Over all real numbers x_1, x_2, x_3, x_4 with $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 9$, find the largest and the smallest value of $2x_1 - x_2 + 2x_3 + 4x_4$, and the numbers $x_1, \dots, x_4$ where each is reached. Justify with the Cauchy-Schwarz inequality. (4 points)

3. A hardware supplier sells three packs. In numbers of (bolts, nuts, washers), pack A holds $\mathbf{a} = (1, 0, 2)$, pack B holds $\mathbf{b} = (0, 1, 1)$ and pack C holds $\mathbf{c} = (1, 1, 0)$. Packs are sold whole and cannot be opened or returned. Diego needs exactly (5, 4, 11), Jenny exactly (2, 6, 1) and Samir exactly (4, 4, 10). For each order, find every list of weights that writes it as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and decide whether it can be filled with packs. (4 points)

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4. Let $\mathbf{v}_1 = (1, -1, 2)$, $\mathbf{v}_2 = (-2, 2, h)$ and $\mathbf{b} = (3, -3, k)$. Find all values of h and k for which $\mathbf{b}$ is (4 points)

(a) a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ with exactly one list of weights (give the weights in terms of h and k);

(b) a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ with more than one list of weights;

(c) not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

5. Let $\mathbf{u}_1 = (1, 0, 1)$, $\mathbf{u}_2 = (0, 1, 2)$, $\mathbf{w}_1 = (1, 1, 3)$, $\mathbf{w}_2 = (2, -1, 0)$, $\mathbf{x}_1 = (1, 2, 5)$ and $\mathbf{x}_2 = (-2, -4, -10)$. (4 points)

(a) Prove that $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$.

(b) Is $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$? Explain.

6. Let $\mathbf{a} = (1, 0, 2, 0, 1)$, $\mathbf{b} = (2, 1, -1, 0, 0)$ and $\mathbf{c} = (1, -2, 0, 1, -1)$ in $\mathbb{R}^5$. (5 points)

(a) Show that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are pairwise orthogonal.

(b) Prove that if $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in $\mathbb{R}^n$ are pairwise orthogonal, then for all scalars p, q, r,

$$\|p\mathbf{x} + q\mathbf{y} + r\mathbf{z}\|^2 = p^2\|\mathbf{x}\|^2 + q^2\|\mathbf{y}\|^2 + r^2\|\mathbf{z}\|^2.$$

(c) Use (b) to find $\|2\mathbf{a} - \mathbf{b} + 3\mathbf{c}\|$.

7. Let $\mathbf{v}_1 = (1, -1, 0, 2)$, $\mathbf{v}_2 = (2, -1, 1, 3)$ and $\mathbf{v}_3 = (0, 1, 1, -1)$ in $\mathbb{R}^4$. (5 points)

(a) Find two equations in x_1, x_2, x_3, x_4 satisfied by exactly the vectors of $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) Find the numbers h and k for which $(1, h, 3, k)$ is in the span, and write that vector as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(c) Is there a vector $\mathbf{w}$ in $\mathbb{R}^4$ for which $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}$ span $\mathbb{R}^4$? Explain.

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8. Let H be the set of vectors in $\mathbb{R}^4$ with $x_1 + x_2 - x_3 + 2x_4 = 0$, and let $\mathbf{a}_1 = (1, 0, 1, 0)$ and $\mathbf{a}_2 = (2, 1, 3, 0)$. (5 points)

(a) Find a basis of H and give $\dim H$.

(b) Prove that a vector $\mathbf{w}$ of H can be added to $\mathbf{a}_1, \mathbf{a}_2$ to give a basis of H exactly when the fourth entry of $\mathbf{w}$ is not 0.

(c) With $\mathbf{w} = (0, 0, 2, 1)$, find the coordinates of (1, 1, 4, 1) relative to the basis $\mathbf{a}_1, \mathbf{a}_2, \mathbf{w}$ of H.

9. Let $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 2, h)$ and $\mathbf{v}_3 = (1, h, 5)$. Find every number h for which $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly dependent, and for each such h give a dependence relation with integer weights. (4 points)

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10. For each real number k, let W_k be the set of vectors (x, y, z) in $\mathbb{R}^3$ with

$$x - 2z + (k-2)y^2 = k^2 - 4.$$

Find every value of k for which W_k is a subspace of $\mathbb{R}^3$. Prove that your values work and that every other value fails, and write W_k as a span for each value that works. (5 points)

11. Let $\mathbf{u}$ be a nonzero vector in $\mathbb{R}^n$. (4 points)

(a) Find every scalar c with $c^2\mathbf{u} = (c+6)\mathbf{u}$. Justify each step from the eight properties and the fact that $c\mathbf{u} = \mathbf{0}$ only when $c = 0$ or $\mathbf{u} = \mathbf{0}$.

(b) Which scalars c satisfy $c^2\mathbf{u} = (c+6)\mathbf{u}$ when $\mathbf{u}$ is instead the zero vector? Explain.

Statements	Reasons

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12. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be a linearly independent list in $\mathbb{R}^n$, let c be a number, and put

$$\mathbf{w} = 2\mathbf{v}_1 - \mathbf{v}_2 + c\,\mathbf{v}_3.$$

(a) Find every value of c for which the list $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}$ is linearly independent, and prove your answer. For any other value, give a dependence relation.

(b) Prove that the list $\mathbf{v}_1, \mathbf{w}, \mathbf{v}_3$ is linearly independent for every c. (4 points)

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