Linear Algebra Unit 1 worksheet: Basis and dimension (answer key)

ANSWERS AT A GLANCE

1. (a) Dimension 1, basis (−1, 2, −3). (b) Dimension 3, basis (1, 1, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1). (c) Dimension 2, basis (1, 1, 0), (0, 1, 1). (d) Dimension 2, basis (1, 0, 1, 0), (0, 1, −1, 2). (e) Dimension 2, basis (−2, 1, 0), (1, 0, 1). (f) Dimension 0, basis the empty list. Any basis of the same subspace is accepted; the dimension is the same for every basis.
2. (a) Neither vector is a multiple of the other, so the list is independent, and it spans V by the definition of V: a basis, and $\dim V = 2$. (b) $\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2$, coordinates (3, 2). (c) No.
3. (a) $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5$, so $\dim V = 3$ (other choices, such as $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5$, are also bases). (b) $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_4 = -2\mathbf{v}_1 + \mathbf{v}_2$. (c) $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_4$ each give a basis of $\mathbb{R}^4$; $\mathbf{e}_1$ does not, because $\mathbf{e}_1$ is in V.
4. (a) (1, 1, 1, 0, 0), (2, −2, 0, 1, 0), (−5, 2, 0, 0, 1), so $\dim N = 3$. (b) Coordinates (4, 2, 1). (c) The third, fourth and fifth entries, the free positions.
5. (a) False: $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ do not span $\mathbb{R}^4$. (b) True, by Result 2 (or Result 6). (c) True: $\dim V = 3$, then Result 6. (d) False: $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (2, 0, 0)$ in the plane $z = 0$. (e) True, by Result 7.
6. See the proof below.

WORKED SOLUTIONS

1. Find the dimension of each subspace and give one basis of it.

(a) $\operatorname{span}\{(2, -4, 6), (-1, 2, -3)\}$ in $\mathbb{R}^3$

(b) $\{(x_1, x_2, x_3, x_4, x_5) : x_1 = x_2 = x_3\}$ in $\mathbb{R}^5$

(c) $\operatorname{span}\{(1, 1, 0), (0, 1, 1), (1, 0, -1)\}$ in $\mathbb{R}^3$

(d) all vectors $(a, b, a - b, 2b)$ for real a and b, in $\mathbb{R}^4$

(e) the vectors $\mathbf{x}$ in $\mathbb{R}^3$ with $\mathbf{x} \cdot (1, 2, -1) = 0$ and $\mathbf{x} \cdot (-2, -4, 2) = 0$

(f) $\{\mathbf{0}\}$ in $\mathbb{R}^6$

ANSWER

(a) Dimension 1, basis (−1, 2, −3). (b) Dimension 3, basis (1, 1, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1). (c) Dimension 2, basis (1, 1, 0), (0, 1, 1). (d) Dimension 2, basis (1, 0, 1, 0), (0, 1, −1, 2). (e) Dimension 2, basis (−2, 1, 0), (1, 0, 1). (f) Dimension 0, basis the empty list. Any basis of the same subspace is accepted; the dimension is the same for every basis.

SOLUTION

(a) (2, −4, 6) = −2(−1, 2, −3), so the span is $\operatorname{span}\{(-1, 2, -3)\}$, a line. One nonzero vector is independent, so it is a basis and the dimension is 1.

(b) The conditions are $x_1 - x_2 = 0$ and $x_2 - x_3 = 0$, with pivots at x_1 and x_2; x_3, x_4, x_5 are free. Then $x_2 = x_3$ and $x_1 = x_3$, so

$$\mathbf{x} = x_3(1, 1, 1, 0, 0) + x_4(0, 0, 0, 1, 0) + x_5(0, 0, 0, 0, 1).$$

By Result 9(b) these three vectors are a basis: dimension 3, five unknowns less two pivots.

(c) (1, 1, 0) − (0, 1, 1) = (1, 0, −1), so the third vector can be deleted without changing the span. The first two are not multiples of each other, so they are independent: a basis, dimension 2, a plane through the origin.

(d) $(a, b, a - b, 2b) = a(1, 0, 1, 0) + b(0, 1, -1, 2)$, so these two vectors span the set. The first entries 1, 0 and second entries 0, 1 show neither is a multiple of the other, so they are independent: dimension 2.

(e) The two conditions are $x + 2y - z = 0$ and $-2x - 4y + 2z = 0$, and the second is −2 times the first, so the set is the plane $x + 2y - z = 0$. With y, z free, $x = -2y + z$, and $(x, y, z) = y(-2, 1, 0) + z(1, 0, 1)$. Dimension 2.

(f) The zero subspace has the empty list as its basis, so its dimension is 0 (it sits in $\mathbb{R}^6$, but dimension counts basis vectors, not entries).

2. Let $V = \operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2\}$ in $\mathbb{R}^4$, where $\mathbf{b}_1 = (1, -3, 2, -3)$ and $\mathbf{b}_2 = (1, -2, -1, -3)$.

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(a) Explain why $\mathbf{b}_1, \mathbf{b}_2$ is a basis of V, and give $\dim V$.

(b) Show that $\mathbf{x} = (5, -13, 4, -15)$ is in V, and find its coordinates relative to $\mathbf{b}_1, \mathbf{b}_2$.

(c) Decide whether $\mathbf{y} = (5, -13, 4, -14)$ is in V.

ANSWER

(a) Neither vector is a multiple of the other, so the list is independent, and it spans V by the definition of V: a basis, and $\dim V = 2$. (b) $\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2$, coordinates (3, 2). (c) No.

SOLUTION

(a) The first entries of $\mathbf{b}_1$ and $\mathbf{b}_2$ are both 1, so if one were a multiple of the other the factor would have to be 1; but the second entries -3 and -2 differ. Neither is a multiple of the other, so the list is independent, and it spans V because V is defined as its span. So it is a basis of V and $\dim V = 2$.

(b) Look for weights with $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = \mathbf{x}$. The first two entries give

$$c_1 + c_2 = 5, \qquad -3c_1 - 2c_2 = -13.$$

Subtract (-3) times the first equation from the second: $c_2 = -13 - (-3)\cdot 5 = 2$. Then $c_1 = 5 - 2 = 3$. The last two entries must agree too:

$$3\cdot 2 + 2\cdot(-1) = 4, \qquad 3\cdot(-3) + 2\cdot(-3) = -15.$$

They do, so $\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2$ is in V, with coordinates (3, 2). By Result 1 no other weights work.

(c) Weights for $\mathbf{y}$ would have to satisfy the same two equations from the first two entries, so they would again be $c_1 = 3$ and $c_2 = 2$. Those weights give fourth entry -15, not -14. So $\mathbf{y}$ is not in V.

3. Let $\mathbf{v}_1 = (1, -1, 2, 0)$, $\mathbf{v}_2 = (2, 1, 1, 3)$, $\mathbf{v}_3 = (3, 0, 3, 3)$, $\mathbf{v}_4 = (0, 3, -3, 3)$, $\mathbf{v}_5 = (0, 1, 0, 2)$, and let $V = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ in $\mathbb{R}^4$.

(a) Choose from the list a basis of V, and give $\dim V$.

(b) Write each vector you left out as a linear combination of your basis.

(c) For each of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$, decide whether adding it to your basis of V gives a basis of $\mathbb{R}^4$.

ANSWER

(a) $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5$, so $\dim V = 3$ (other choices, such as $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5$, are also bases). (b) $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_4 = -2\mathbf{v}_1 + \mathbf{v}_2$. (c) $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_4$ each give a basis of $\mathbb{R}^4$; $\mathbf{e}_1$ does not, because $\mathbf{e}_1$ is in V.

SOLUTION

(a) By Result 9(a), reduce the matrix with these columns. $R_2 + R_1$ and $R_3 - 2R_1$, then $R_3 + R_2$ and $R_4 - R_2$, then $R_4 - R_3$:

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ -1 & 1 & 0 & 3 & 1 \\ 2 & 1 & 3 & -3 & 0 \\ 0 & 3 & 3 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 3 & 3 & 3 & 1 \\ 0 & -3 & -3 & -3 & 0 \\ 0 & 3 & 3 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 3 & 3 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The pivots are in columns 1, 2 and 5, so $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5$ is a basis of V and $\dim V = 3$.

(b) Weight 1 on column 3 and 0 on column 4: row 3 gives $c_5 = 0$, row 2 gives $3c_2 + 3 = 0$, so $c_2 = -1$, and row 1 gives $c_1 - 2 + 3 = 0$, so $c_1 = -1$. So $-\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$, that is, $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$. Weight 1 on column 4 and 0 on column 3: $c_5 = 0$, $c_2 = -1$, and $c_1 - 2 = 0$, so $c_1 = 2$; that is, $\mathbf{v}_4 = -2\mathbf{v}_1 + \mathbf{v}_2$. Check: $-2(1, -1, 2, 0) + (2, 1, 1, 3) = (0, 3, -3, 3)$.

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(c) First find which vectors are in V. Reduce $[\,\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_5 \mid (x_1,x_2,x_3,x_4)\,]$ with the same steps:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & x_1 \\ -1 & 1 & 1 & x_2 \\ 2 & 1 & 0 & x_3 \\ 0 & 3 & 2 & x_4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & x_1 \\ 0 & 3 & 1 & x_1+x_2 \\ 0 & 0 & 1 & -x_1+x_2+x_3 \\ 0 & 0 & 0 & -2x_2-x_3+x_4 \end{array}\right].$$

So a vector is in V exactly when $2x_2 + x_3 - x_4 = 0$. For $\mathbf{e}_1$ this is 0 = 0: indeed $\mathbf{e}_1 = -\frac{1}{3}\mathbf{v}_1 + \frac{2}{3}\mathbf{v}_2 - \mathbf{v}_5$, so adding $\mathbf{e}_1$ gives a dependent list, not a basis. For $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ the left side is 2, 1, −1, so none of them is in $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5\}$. By the lemma for growing a list, adding any one of them gives four independent vectors in $\mathbb{R}^4$, which is a basis of $\mathbb{R}^4$ by Result 6. This does not depend on which basis of V was chosen in (a): every basis of V has span V.

4. Let N be the set of solutions in $\mathbb{R}^5$ of the system

$$\begin{aligned} x_1 + x_2 - 2x_3 + 3x_5 &= 0 \\ x_1 + 2x_2 - 3x_3 + 2x_4 + x_5 &= 0. \end{aligned}$$

(a) Find a basis of N with one vector for each free variable, and give $\dim N$.

(b) Show that $\mathbf{x} = (3,2,4,2,1)$ is in N, and find its coordinates relative to your basis.

(c) Explain why, relative to this basis, the coordinates of every vector of N can be read off from three of its entries, and say which three.

ANSWER

(a) (1, 1, 1, 0, 0), (2, −2, 0, 1, 0), (−5, 2, 0, 0, 1), so $\dim N = 3$. (b) Coordinates (4, 2, 1). (c) The third, fourth and fifth entries, the free positions.

SOLUTION

(a) Subtracting the first equation from the second gives $x_2 - x_3 + 2x_4 - 2x_5 = 0$. The pivots are at x_1 and x_2, so x_3, x_4, x_5 are free. Then $x_2 = x_3 - 2x_4 + 2x_5$, and the first equation gives

$$x_1 = -x_2 + 2x_3 - 3x_5 = x_3 + 2x_4 - 5x_5.$$

So

$$\mathbf{x} = x_3\begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}2\\-2\\0\\1\\0\end{bmatrix} + x_5\begin{bmatrix}-5\\2\\0\\0\\1\end{bmatrix},$$

and by Result 9(b) the three vectors $\mathbf{s}_3 = (1,1,1,0,0)$, $\mathbf{s}_4 = (2,-2,0,1,0)$, $\mathbf{s}_5 = (-5,2,0,0,1)$ are a basis of N. So $\dim N = 3$.

(b) 3 + 2 − 8 + 3 = 0 and 3 + 4 − 12 + 4 + 1 = 0, so $\mathbf{x}$ is in N. Its free entries are $x_3 = 4$, $x_4 = 2$, $x_5 = 1$, and

$$4\mathbf{s}_3 + 2\mathbf{s}_4 + \mathbf{s}_5 = (4+4-5,\ 4-4+2,\ 4,\ 2,\ 1) = (3,2,4,2,1).$$

The coordinates are (4, 2, 1), and by Result 1 they are the only ones.

(c) Each basis vector has 1 at its own free position and 0 at the other two free positions. So in $c_3\mathbf{s}_3 + c_4\mathbf{s}_4 + c_5\mathbf{s}_5$ the third entry is c_3, the fourth is c_4 and the fifth is c_5. The coordinates of a vector of N are its

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third, fourth and fifth entries.

5. Decide whether each statement is true or false. For a true statement, name the result that gives it; for a false one, give a counterexample with actual vectors.

(a) Any three linearly independent vectors in $\mathbb{R}^4$ span $\mathbb{R}^4$.

(b) If V is a subspace of $\mathbb{R}^5$ with $\dim V = 3$, then any four vectors in V are linearly dependent.

(c) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ span a subspace V, and V contains some linearly independent list of three vectors, then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is a basis of V.

(d) If $U = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ lies inside a plane P through the origin in $\mathbb{R}^3$, then $U = P$.

(e) If V is a subspace of $\mathbb{R}^6$ with $\dim V = 6$, then $V = \mathbb{R}^6$.

ANSWER

(a) False: $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ do not span $\mathbb{R}^4$. (b) True, by Result 2 (or Result 6). (c) True: $\dim V = 3$, then Result 6. (d) False: $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (2, 0, 0)$ in the plane $z = 0$. (e) True, by Result 7.

SOLUTION

(a) False. $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in $\mathbb{R}^4$ are independent, but every combination of them has fourth entry 0, so $\mathbf{e}_4$ is not in their span. ($\dim \mathbb{R}^4 = 4$, so Result 6 needs four vectors.)

(b) True. V is spanned by a basis of 3 vectors, so by Result 2 an independent list in V has at most 3 vectors; four vectors in V are dependent. This is the "more than k" part of Result 6.

(c) True. Shrinking $\mathbf{u}, \mathbf{v}, \mathbf{w}$ gives a basis of V with at most 3 vectors (Result 5), so $\dim V \leq 3$. A basis of V spans V, and V holds three independent vectors, so by Result 2 the basis has at least 3 vectors. So $\dim V = 3$, and three vectors that span a space of dimension 3 are independent by Result 6: a basis.

(d) False. Let P be the plane $z = 0$, $\mathbf{u}_1 = (1, 0, 0)$ and $\mathbf{u}_2 = (2, 0, 0)$. Both lie in P, but $\mathbf{u}_2 = 2\mathbf{u}_1$, so U is the line $\text{span}\{(1, 0, 0)\}$, of dimension 1, while $\dim P = 2$; (0, 1, 0) is in P and not in U. Result 7 needs equal dimensions, and two vectors need not give dimension 2.

(e) True. $V \subseteq \mathbb{R}^6$ and $\dim V = 6 = \dim \mathbb{R}^6$, so $V = \mathbb{R}^6$ by Result 7.

6. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be a basis of a subspace V of $\mathbb{R}^n$, and let

$$\mathbf{w}_1 = \mathbf{v}_1, \qquad \mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2, \qquad \mathbf{w}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3.$$

(a) Prove that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ is a basis of V. Check only that the list lies in V and is independent, and say why it also spans V.

(b) Find the coordinates of $\mathbf{v}_3$, and of $\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3$, relative to $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.

ANSWER

(a) Each $\mathbf{w}_i$ is a combination of vectors of V, so it is in V; a zero combination forces all weights 0; three independent vectors in V, with $\dim V = 3$, span V by Result 6. (b) $\mathbf{v}_3$: (0, −1, 1). $\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3$: (−1, −1, 3).

SOLUTION

(a) *In V.* V is a subspace, so it contains every combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$; each $\mathbf{w}_i$ is one.

Independent. Suppose $a\mathbf{w}_1 + b\mathbf{w}_2 + c\mathbf{w}_3 = \mathbf{0}$. Collecting the weights on each $\mathbf{v}_i$,

$$(a + b + c)\mathbf{v}_1 + (b + c)\mathbf{v}_2 + c\,\mathbf{v}_3 = \mathbf{0}.$$

The $\mathbf{v}_i$ are independent, so $a + b + c = 0$, $b + c = 0$ and $c = 0$. Then $b = 0$ and $a = 0$: only the trivial combination gives $\mathbf{0}$.

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Spans. V has a basis of three vectors, so $\dim V = 3$. Three independent vectors in a subspace of dimension 3 span it (Result 6). So $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ is a basis of V.

(b) $\mathbf{w}_3 - \mathbf{w}_2 = \mathbf{v}_3$, so $\mathbf{v}_3 = 0\mathbf{w}_1 - \mathbf{w}_2 + \mathbf{w}_3$, with coordinates (0, −1, 1).

For $\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3 = a\mathbf{w}_1 + b\mathbf{w}_2 + c\mathbf{w}_3$, use the expansion above and match weights, which are unique because the $\mathbf{v}_i$ are a basis (Result 1): $c = 3$, $b + c = 2$, $a + b + c = 1$. So $c = 3$, $b = -1$, $a = -1$, and the coordinates are (−1, −1, 3). Check: $-\mathbf{v}_1 - (\mathbf{v}_1 + \mathbf{v}_2) + 3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = \mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3$.

SCORING

Homework: (a) needs membership in V, the expansion in the $\mathbf{v}_i$ with the independence of the $\mathbf{v}_i$ used to set each weight to 0, and spanning from $\dim V = 3$ and Result 6 (not a separate spanning computation); (b) needs both coordinate lists.

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