Linear Algebra Unit 1 worksheet: Linear combination (answer key)

ANSWERS AT A GLANCE

1. $(0, 29, 9, 6, -19)$
2. $\mathbf{b} = 4\mathbf{v}_1 - 2\mathbf{v}_2$, with exactly one list of weights.
3. (a) $c_1 = x_1 - x_2$, $c_2 = x_2 - x_3$, $c_3 = x_3 - x_4$, $c_4 = x_4$, and these are the only weights. (b) $(5, -2, 0, 3) = 7\mathbf{f}_1 - 2\mathbf{f}_2 - 3\mathbf{f}_3 + 3\mathbf{f}_4$. (c) Exactly the vectors with $x_1 > x_2 > x_3 > x_4 > 0$.
4. (a) $\mathbf{u} = \frac{1}{7}\mathbf{x} + \frac{2}{7}\mathbf{y}$ and $\mathbf{v} = \frac{3}{7}\mathbf{x} - \frac{1}{7}\mathbf{y}$. (b) $4\mathbf{u} + \mathbf{v} = \mathbf{x} + \mathbf{y}$.
5. (a) $c_1 = 3 - 2t$, $c_2 = 2 + t$, $c_3 = t$, for any real number t. (b) $t = \frac{3}{2}$, so $\mathbf{b} = \frac{7}{2}\mathbf{v}_2 + \frac{3}{2}\mathbf{v}_3$.
6. (a) $(5, 8) = \mathbf{a} + 2\mathbf{b}$: one **a** and two **b** stitches forward. $(5, 1) = 2\mathbf{a} - \mathbf{b}$: two **a** stitches forward and one **b** stitch backward. $(4, 3)$ cannot be reached: its only weights are $\frac{9}{7}$ and $\frac{1}{7}$. (b) The weights for (x, y) are $c_1 = \frac{3x - y}{7}$ and $c_2 = \frac{3y - 2x}{7}$; they are whole numbers exactly when 7 divides $3x - y$.
7. See the proof below.

WORKED SOLUTIONS

1. Let $\mathbf{u} = (-4, 3, 6, 6, -5)$, $\mathbf{v} = (-2, -5, 1, 6, 1)$ and $\mathbf{w} = (1, 4, 0, 6, -3)$ in $\mathbb{R}^5$. Find the linear combination $2\mathbf{u} - 3\mathbf{v} + 2\mathbf{w}$.

ANSWER

$(0, 29, 9, 6, -19)$

SOLUTION

Scale each vector by its weight, then add position by position:

$$\begin{aligned} 2\mathbf{u} - 3\mathbf{v} + 2\mathbf{w} &= (-8, 6, 12, 12, -10) \\ &\quad + (6, 15, -3, -18, -3) \\ &\quad + (2, 8, 0, 12, -6) \\ &= (0, 29, 9, 6, -19). \end{aligned}$$

2. Let $\mathbf{v}_1 = (-2, 5, -3, 0)$ and $\mathbf{v}_2 = (-3, 3, 3, 3)$ in $\mathbb{R}^4$. Show that $\mathbf{b} = (-2, 14, -18, -6)$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, and find the weights.

ANSWER

$\mathbf{b} = 4\mathbf{v}_1 - 2\mathbf{v}_2$, with exactly one list of weights.

SOLUTION

We need $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{b}$: one equation per entry, four equations in the two unknowns c_1, c_2, with augmented matrix

$$\left[\begin{array}{cc|c} -2 & -3 & -2 \\ 5 & 3 & 14 \\ -3 & 3 & -18 \\ 0 & 3 & -6 \end{array}\right]$$

Eliminating in the first two equations allows only $c_1 = 4$ and $c_2 = -2$. The last two equations must hold for these values or for nothing, and they hold: $-3 \cdot 4 + 3 \cdot (-2) = -18$ and $0 \cdot 4 + 3 \cdot (-2) = -6$.

So $\mathbf{b} = 4\mathbf{v}_1 - 2\mathbf{v}_2$, and no other weights work. Check: $(-8, 20, -12, 0) + (6, -6, -6, -6) = (-2, 14, -18, -6)$.

3. Let $\mathbf{f}_1 = (1, 0, 0, 0)$, $\mathbf{f}_2 = (1, 1, 0, 0)$, $\mathbf{f}_3 = (1, 1, 1, 0)$ and $\mathbf{f}_4 = (1, 1, 1, 1)$ in $\mathbb{R}^4$.

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(a) Find the weights that write any $\mathbf{x} = (x_1, x_2, x_3, x_4)$ as $c_1\mathbf{f}_1 + c_2\mathbf{f}_2 + c_3\mathbf{f}_3 + c_4\mathbf{f}_4$, and explain why no other weights work.

(b) Write (5, −2, 0, 3) as a combination of $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4$.

(c) Which vectors $\mathbf{x}$ can be written with all four weights positive?

ANSWER

(a) $c_1 = x_1 - x_2$, $c_2 = x_2 - x_3$, $c_3 = x_3 - x_4$, $c_4 = x_4$, and these are the only weights. (b) $(5, -2, 0, 3) = 7\mathbf{f}_1 - 2\mathbf{f}_2 - 3\mathbf{f}_3 + 3\mathbf{f}_4$. (c) Exactly the vectors with $x_1 > x_2 > x_3 > x_4 > 0$.

SOLUTION

(a) Scaling and adding,

$$c_1\mathbf{f}_1 + c_2\mathbf{f}_2 + c_3\mathbf{f}_3 + c_4\mathbf{f}_4 = (c_1 + c_2 + c_3 + c_4,\ c_2 + c_3 + c_4,\ c_3 + c_4,\ c_4).$$

Setting this equal to $\mathbf{x}$ gives four equations. Work from the bottom: the fourth says $c_4 = x_4$; the third minus the fourth gives $c_3 = x_3 - x_4$; the second minus the third gives $c_2 = x_2 - x_3$; the first minus the second gives $c_1 = x_1 - x_2$. Every step was forced, so no other weights work, and these do work: the sums telescope, for example $c_2 + c_3 + c_4 = (x_2 - x_3) + (x_3 - x_4) + x_4 = x_2$.

(b) $c_1 = 5 - (-2) = 7$, $c_2 = -2 - 0 = -2$, $c_3 = 0 - 3 = -3$, $c_4 = 3$, so

$$(5, -2, 0, 3) = 7\mathbf{f}_1 - 2\mathbf{f}_2 - 3\mathbf{f}_3 + 3\mathbf{f}_4.$$

Check: (7 − 2 − 3 + 3, −2 − 3 + 3, −3 + 3, 3) = (5, −2, 0, 3).

(c) The weights are unique, so all four are positive exactly when $x_1 - x_2 > 0$, $x_2 - x_3 > 0$, $x_3 - x_4 > 0$ and $x_4 > 0$, that is, when $x_1 > x_2 > x_3 > x_4 > 0$. For instance (5, −2, 0, 3) fails, as its weights in (b) show.

4. Vectors $\mathbf{u}$ and $\mathbf{v}$ in some $\mathbb{R}^n$ are used to build $\mathbf{x} = \mathbf{u} + 2\mathbf{v}$ and $\mathbf{y} = 3\mathbf{u} - \mathbf{v}$.

(a) Write $\mathbf{u}$ and $\mathbf{v}$ as linear combinations of $\mathbf{x}$ and $\mathbf{y}$.

(b) Write $4\mathbf{u} + \mathbf{v}$ as a linear combination of $\mathbf{x}$ and $\mathbf{y}$. Check your answer by writing it back in terms of $\mathbf{u}$ and $\mathbf{v}$.

ANSWER

(a) $\mathbf{u} = \frac{1}{7}\mathbf{x} + \frac{2}{7}\mathbf{y}$ and $\mathbf{v} = \frac{3}{7}\mathbf{x} - \frac{1}{7}\mathbf{y}$. (b) $4\mathbf{u} + \mathbf{v} = \mathbf{x} + \mathbf{y}$.

SOLUTION

(a) Eliminate $\mathbf{v}$ as in a system of two equations: $\mathbf{x} + 2\mathbf{y} = \mathbf{u} + 2\mathbf{v} + 6\mathbf{u} - 2\mathbf{v} = 7\mathbf{u}$, so $\mathbf{u} = \frac{1}{7}\mathbf{x} + \frac{2}{7}\mathbf{y}$. Eliminate $\mathbf{u}$: $3\mathbf{x} - \mathbf{y} = 3\mathbf{u} + 6\mathbf{v} - 3\mathbf{u} + \mathbf{v} = 7\mathbf{v}$, so $\mathbf{v} = \frac{3}{7}\mathbf{x} - \frac{1}{7}\mathbf{y}$.

(b) A combination of combinations is a combination: collect the weights on $\mathbf{x}$ and on $\mathbf{y}$.

$$4\mathbf{u} + \mathbf{v} = \left(\tfrac{4}{7} + \tfrac{3}{7}\right)\mathbf{x} + \left(\tfrac{8}{7} - \tfrac{1}{7}\right)\mathbf{y} = \mathbf{x} + \mathbf{y}.$$

Check: $\mathbf{x} + \mathbf{y} = (\mathbf{u} + 2\mathbf{v}) + (3\mathbf{u} - \mathbf{v}) = 4\mathbf{u} + \mathbf{v}$.

5. Let $\mathbf{v}_1 = (1, 0, 2, -1)$, $\mathbf{v}_2 = (0, 1, 1, 3)$, $\mathbf{v}_3 = (2, -1, 3, -5)$ and $\mathbf{b} = (3, 2, 8, 3)$.

(a) Find every list of weights c_1, c_2, c_3 with $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$.

(b) Which list in (a) has $c_1 = 0$? Use it to write $\mathbf{b}$ as a combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ alone.

ANSWER

(a) $c_1 = 3 - 2t$, $c_2 = 2 + t$, $c_3 = t$, for any real number t. (b) $t = \frac{3}{2}$, so $\mathbf{b} = \frac{7}{2}\mathbf{v}_2 + \frac{3}{2}\mathbf{v}_3$.

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SOLUTION

(a) The augmented matrix has columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{b}$. Subtract 2 times row 1 from row 3 and add row 1 to row 4:

$$\left[\begin{array}{rrr|r} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 2 \\ 2 & 1 & 3 & 8 \\ -1 & 3 & -5 & 3 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 3 & -3 & 6 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(the last step subtracts row 2 from row 3 and 3 times row 2 from row 4). Two equations are left, $c_1 + 2c_3 = 3$ and $c_2 - c_3 = 2$, and c_3 is free. With $c_3 = t$:

$$c_1 = 3 - 2t, \qquad c_2 = 2 + t, \qquad c_3 = t.$$

Check with $t = 0$: $3\mathbf{v}_1 + 2\mathbf{v}_2 = (3, 0, 6, -3) + (0, 2, 2, 6) = (3, 2, 8, 3)$.

(b) $c_1 = 0$ means $3 - 2t = 0$, so $t = \dfrac{3}{2}$, $c_2 = \dfrac{7}{2}$ and $c_3 = \dfrac{3}{2}$:

$$\mathbf{b} = \tfrac{7}{2}\mathbf{v}_2 + \tfrac{3}{2}\mathbf{v}_3.$$

Check: $\left(0, \dfrac{7}{2}, \dfrac{7}{2}, \dfrac{21}{2}\right) + \left(3, -\dfrac{3}{2}, \dfrac{9}{2}, -\dfrac{15}{2}\right) = (3, 2, 8, 3)$.

6. Mei's embroidery machine moves its needle by whole stitches of two kinds, $\mathbf{a} = (3, 2)$ and $\mathbf{b} = (1, 3)$, in millimeters (across, up). It can sew either stitch forward or backward (a backward stitch moves by $-\mathbf{a}$ or $-\mathbf{b}$), but never part of a stitch. The needle starts at the origin, and the order of the stitches does not matter.

(a) For each of the points (5, 8), (5, 1) and (4, 3), decide whether the needle can land on it, and if so say which stitches to sew.

(b) Show that the needle can land on a point (x, y) with whole-number coordinates exactly when $3x - y$ is a multiple of 7.

ANSWER

(a) $(5, 8) = \mathbf{a} + 2\mathbf{b}$: one $\mathbf{a}$ and two $\mathbf{b}$ stitches forward. $(5, 1) = 2\mathbf{a} - \mathbf{b}$: two $\mathbf{a}$ stitches forward and one $\mathbf{b}$ stitch backward. (4, 3) cannot be reached: its only weights are $\dfrac{9}{7}$ and $\dfrac{1}{7}$. (b) The weights for (x, y) are $c_1 = \dfrac{3x - y}{7}$ and $c_2 = \dfrac{3y - 2x}{7}$; they are whole numbers exactly when 7 divides $3x - y$.

SOLUTION

A landing point is $c_1\mathbf{a} + c_2\mathbf{b}$ with whole numbers c_1, c_2 (negative for backward). Matching entries, $3c_1 + c_2 = x$ and $2c_1 + 3c_2 = y$. Three times the first minus the second gives $7c_1 = 3x - y$; three times the second minus twice the first gives $7c_2 = 3y - 2x$. So every point has exactly one list of weights,

$$c_1 = \frac{3x - y}{7}, \qquad c_2 = \frac{3y - 2x}{7},$$

and the question is whether both are whole numbers.

(a) (5, 8): $c_1 = \dfrac{15 - 8}{7} = 1$, $c_2 = \dfrac{24 - 10}{7} = 2$. Check: (3, 2) + (2, 6) = (5, 8). (5, 1): $c_1 = \dfrac{15 - 1}{7} = 2$, $c_2 = \dfrac{3 - 10}{7} = -1$. Check: (6, 4) – (1, 3) = (5, 1). (4, 3): $c_1 = \dfrac{12 - 3}{7} = \dfrac{9}{7}$, $c_2 = \dfrac{9 - 8}{7} = \dfrac{1}{7}$. These are the only weights, and they are not whole numbers, so the needle cannot land there.

(b) If the needle lands on (x, y), then c_1 is a whole number and $3x - y = 7c_1$ is a multiple of 7. Conversely, if $3x - y = 7m$ for a whole number m, then $c_1 = m$, and $y = 3x - 7m$ gives $3y - 2x = 7x - 21m =$

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$7(x - 3m)$, so $c_2 = x - 3m$ is a whole number too.

SCORING

Homework: (a) needs the weights for all three points with a reason (4, 3) fails; (b) needs both directions (whole weights give a multiple of 7, and a multiple of 7 gives whole weights for both c_1 and c_2).

7. Let $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{b}$ be vectors in $\mathbb{R}^n$ with $\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

(a) Prove that if $c_1 \neq 0$, then $\mathbf{v}_1$ is a linear combination of $\mathbf{b}$ and $\mathbf{v}_2$.

(b) Give actual vectors in $\mathbb{R}^2$ showing that when $c_1 = 0$, $\mathbf{v}_1$ need not be a linear combination of $\mathbf{b}$ and $\mathbf{v}_2$.

ANSWER

(a) $\mathbf{v}_1 = \dfrac{1}{c_1}\mathbf{b} - \dfrac{c_2}{c_1}\mathbf{v}_2$. (b) For example $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (0, 1)$ and $\mathbf{b} = 2\mathbf{v}_2 = (0, 2)$.

SOLUTION

(a) Subtract $c_2\mathbf{v}_2$ from both sides: $c_1\mathbf{v}_1 = \mathbf{b} - c_2\mathbf{v}_2$. Since $c_1 \neq 0$, multiply both sides by $\dfrac{1}{c_1}$:

$$\mathbf{v}_1 = \frac{1}{c_1}\mathbf{b} - \frac{c_2}{c_1}\mathbf{v}_2,$$

a linear combination of $\mathbf{b}$ and $\mathbf{v}_2$ with weights $\dfrac{1}{c_1}$ and $-\dfrac{c_2}{c_1}$.

(b) Take $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (0, 1)$ and $\mathbf{b} = 0\mathbf{v}_1 + 2\mathbf{v}_2 = (0, 2)$. Every combination $r\mathbf{b} + s\mathbf{v}_2 = (0, 2r + s)$ has first entry 0, so none equals $(1, 0) = \mathbf{v}_1$.

SCORING

Homework: (a) needs the step that uses $c_1 \neq 0$ (dividing by c_1) and the weights named; (b) needs specific vectors with $c_1 = 0$ and a reason no combination works.

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