Linear Algebra Unit 1 worksheet: Linear independence (answer key)

ANSWERS AT A GLANCE

1. (a) $3\mathbf{a}+\mathbf{b}+0\mathbf{c}=\mathbf{0}$; $\mathbf{a}$ and $\mathbf{b}$ are combinations of the others, $\mathbf{c}$ is not. (b) $0\mathbf{p}+1\mathbf{q}+0\mathbf{r}=\mathbf{0}$; only $\mathbf{q}$ is a combination of the others. (c) $2\mathbf{x}+\mathbf{y}-\mathbf{z}=\mathbf{0}$; all three are. (d) $\mathbf{d}_1+\mathbf{d}_2-\mathbf{d}_3+0\mathbf{d}_4=\mathbf{0}$; $\mathbf{d}_1$, $\mathbf{d}_2$ and $\mathbf{d}_3$ are, $\mathbf{d}_4$ is not. (Any nonzero multiple of each relation is also right.)
2. (a) $\mathbf{v}_1-2\mathbf{v}_2-\mathbf{v}_3=\mathbf{0}$. (b) $\mathbf{v}_1=2\mathbf{v}_2+\mathbf{v}_3$. (c) $\mathbf{b}=3\mathbf{v}_2+5\mathbf{v}_3$. (d) $\mathbf{b}=(2+t)\mathbf{v}_1+(-1-2t)\mathbf{v}_2+(3-t)\mathbf{v}_3$ for every real t.
3. (a) $c_1=-2s-t$, $c_2=s-t$, $c_3=s$, $c_4=t$, $c_5=t$, for real s,t not both 0. (b) $-3\mathbf{v}_2-\mathbf{v}_3+2\mathbf{v}_4+2\mathbf{v}_5=\mathbf{0}$. (c) All five.
4. The dependence relations are $c_1=-2t$, $c_2=-4t$, $c_3=t$ for any $t\neq 0$. With $t=-1$, $2\mathbf{v}_1+4\mathbf{v}_2-\mathbf{v}_3=\mathbf{0}$, so $\mathbf{v}_1=\frac{1}{2}(\mathbf{v}_3-4\mathbf{v}_2)$.
5. See the proof below.

WORKED SOLUTIONS

1. Each list below is linearly dependent. For each one, give a dependence relation with integer weights, and say which vectors of the list are linear combinations of the others. Do not set up a system.

(a) $\mathbf{a}=(2,-1,3)$, $\mathbf{b}=(-6,3,-9)$, $\mathbf{c}=(1,1,1)$

(b) $\mathbf{p}=(1,0,2,-1)$, $\mathbf{q}=(0,0,0,0)$, $\mathbf{r}=(3,1,1,1)$

(c) $\mathbf{x}=(2,-1)$, $\mathbf{y}=(1,3)$, $\mathbf{z}=(5,1)$

(d) $\mathbf{d}_1=(1,1,0,0)$, $\mathbf{d}_2=(0,1,1,0)$, $\mathbf{d}_3=(1,2,1,0)$, $\mathbf{d}_4=(0,0,0,1)$

ANSWER

(a) $3\mathbf{a}+\mathbf{b}+0\mathbf{c}=\mathbf{0}$; $\mathbf{a}$ and $\mathbf{b}$ are combinations of the others, $\mathbf{c}$ is not. (b) $0\mathbf{p}+1\mathbf{q}+0\mathbf{r}=\mathbf{0}$; only $\mathbf{q}$ is a combination of the others. (c) $2\mathbf{x}+\mathbf{y}-\mathbf{z}=\mathbf{0}$; all three are. (d) $\mathbf{d}_1+\mathbf{d}_2-\mathbf{d}_3+0\mathbf{d}_4=\mathbf{0}$; $\mathbf{d}_1$, $\mathbf{d}_2$ and $\mathbf{d}_3$ are, $\mathbf{d}_4$ is not. (Any nonzero multiple of each relation is also right.)

SOLUTION

(a) $\mathbf{b}=-3\mathbf{a}$, so $3\mathbf{a}+\mathbf{b}+0\mathbf{c}=\mathbf{0}$. Then $\mathbf{b}=-3\mathbf{a}+0\mathbf{c}$ and $\mathbf{a}=-\frac{1}{3}\mathbf{b}+0\mathbf{c}$. Every combination of $\mathbf{a}$ and $\mathbf{b}$ is a multiple of $\mathbf{a}$, and (1, 1, 1) is not a multiple of (2, −1, 3) (the factor would be $\frac{1}{2}$ from the first entries and −1 from the second), so $\mathbf{c}$ is not a combination of the others.

(b) Weight 1 on the zero vector: $0\mathbf{p}+1\mathbf{q}+0\mathbf{r}=\mathbf{0}$. Also $\mathbf{q}=0\mathbf{p}+0\mathbf{r}$. A combination of $\mathbf{q}$ and $\mathbf{r}$ is a multiple of $\mathbf{r}$, and $\mathbf{p}$ is not one (the second entry of $\mathbf{p}$ is 0 but the first is not, while $\mathbf{r}$ has second entry 1). The same comparison shows $\mathbf{r}$ is not a multiple of $\mathbf{p}$. So only $\mathbf{q}$ is a combination of the others.

(c) $2\mathbf{x}+\mathbf{y}=(4,-2)+(1,3)=(5,1)=\mathbf{z}$, so $2\mathbf{x}+\mathbf{y}-\mathbf{z}=\mathbf{0}$. Every weight is nonzero, so each vector can be solved for: $\mathbf{x}=\frac{1}{2}(\mathbf{z}-\mathbf{y})$, $\mathbf{y}=\mathbf{z}-2\mathbf{x}$, $\mathbf{z}=2\mathbf{x}+\mathbf{y}$.

(d) $\mathbf{d}_1+\mathbf{d}_2=(1,2,1,0)=\mathbf{d}_3$, so $\mathbf{d}_1+\mathbf{d}_2-\mathbf{d}_3+0\mathbf{d}_4=\mathbf{0}$, and $\mathbf{d}_1=\mathbf{d}_3-\mathbf{d}_2$, $\mathbf{d}_2=\mathbf{d}_3-\mathbf{d}_1$, $\mathbf{d}_3=\mathbf{d}_1+\mathbf{d}_2$. Every combination of $\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3$ has last entry 0, while $\mathbf{d}_4$ has last entry 1, so $\mathbf{d}_4$ is not a combination of the others. A dependent list need not have every vector redundant.

2. Vectors $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ and $\mathbf{b}$ in some $\mathbb{R}^n$ satisfy both

$$\mathbf{b}=2\mathbf{v}_1-\mathbf{v}_2+3\mathbf{v}_3 \quad\text{and}\quad \mathbf{b}=\mathbf{v}_1+\mathbf{v}_2+4\mathbf{v}_3.$$

(a) Find a dependence relation among $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$.

(b) Write $\mathbf{v}_1$ as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$.

(c) Write $\mathbf{b}$ as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ alone.

(d) Explain why $\mathbf{b}$ has infinitely many lists of weights as a combination of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$, and give infinitely many such lists in terms of one number t.

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ANSWER

(a) $\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$. (b) $\mathbf{v}_1 = 2\mathbf{v}_2 + \mathbf{v}_3$. (c) $\mathbf{b} = 3\mathbf{v}_2 + 5\mathbf{v}_3$. (d) $\mathbf{b} = (2+t)\mathbf{v}_1 + (-1-2t)\mathbf{v}_2 + (3-t)\mathbf{v}_3$ for every real t.

SOLUTION

(a) Subtract the second equation from the first: $\mathbf{0} = (2-1)\mathbf{v}_1 + (-1-1)\mathbf{v}_2 + (3-4)\mathbf{v}_3$, so

$$\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}.$$

The weights are not all zero, so this is a dependence relation (as Result 6 says: two different lists of weights force one).

(b) The weight on $\mathbf{v}_1$ is $1 \neq 0$, so solve for it: $\mathbf{v}_1 = 2\mathbf{v}_2 + \mathbf{v}_3$.

(c) Substitute (b) into the first equation: $\mathbf{b} = 2(2\mathbf{v}_2 + \mathbf{v}_3) - \mathbf{v}_2 + 3\mathbf{v}_3 = 3\mathbf{v}_2 + 5\mathbf{v}_3$. (The second equation gives the same: $(2\mathbf{v}_2 + \mathbf{v}_3) + \mathbf{v}_2 + 4\mathbf{v}_3$.)

(d) Add t times the relation in (a) to the first equation; adding $\mathbf{0}$ changes nothing:

$$\mathbf{b} = (2+t)\mathbf{v}_1 + (-1-2t)\mathbf{v}_2 + (3-t)\mathbf{v}_3.$$

Different t give different first weights, so these are infinitely many different lists. $t = -1$ gives the second equation and $t = -2$ gives (c). (If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ have other relations too, there are still more lists; the question asks only for infinitely many.)

3. Let $\mathbf{v}_1 = (1, 2, 0, -1)$, $\mathbf{v}_2 = (2, 3, 1, 0)$, $\mathbf{v}_3 = (0, 1, -1, -2)$, $\mathbf{v}_4 = (0, 1, -1, 3)$ and $\mathbf{v}_5 = (3, 4, 2, -4)$ in $\mathbb{R}^4$.

(a) Row reduce and find every dependence relation $c_1\mathbf{v}_1 + \cdots + c_5\mathbf{v}_5 = \mathbf{0}$.

(b) Find the dependence relation with $c_1 = 0$ and $c_5 = 2$.

(c) Which of the five vectors are linear combinations of the other four? Explain.

ANSWER

(a) $c_1 = -2s - t$, $c_2 = s - t$, $c_3 = s$, $c_4 = t$, $c_5 = t$, for real s, t not both 0. (b) $-3\mathbf{v}_2 - \mathbf{v}_3 + 2\mathbf{v}_4 + 2\mathbf{v}_5 = \mathbf{0}$. (c) All five.

SOLUTION

(a) The matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_5$; first $R_2 - 2R_1$ and $R_4 + R_1$, then $R_3 + R_2$ and $R_4 + 2R_2$, then swap rows 3 and 4:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 2 & 3 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 & 2 \\ -1 & 0 & -2 & 3 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & -1 & 1 & 1 & -2 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 2 & -2 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & -1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 5 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivots sit in columns 1, 2 and 4, so $c_3 = s$ and $c_5 = t$ are free. Row 3: $5c_4 - 5t = 0$, so $c_4 = t$. Row 2: $-c_2 + s + t - 2t = 0$, so $c_2 = s - t$. Row 1: $c_1 + 2(s-t) + 3t = 0$, so $c_1 = -2s - t$. The relations are

$$(-2s - t)\mathbf{v}_1 + (s - t)\mathbf{v}_2 + s\,\mathbf{v}_3 + t\,\mathbf{v}_4 + t\,\mathbf{v}_5 = \mathbf{0},$$

with s and t not both 0. For example $s = 1, t = 0$ gives $-2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$, and $s = 0, t = 1$ gives $-\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_4 + \mathbf{v}_5 = \mathbf{0}$.

(b) $c_5 = t = 2$ and $c_1 = -2s - 2 = 0$ give $s = -1$, so the weights are (0, −3, −1, 2, 2):

$$-3\mathbf{v}_2 - \mathbf{v}_3 + 2\mathbf{v}_4 + 2\mathbf{v}_5 = \mathbf{0}.$$

Check: (−6, −9, −3, 0) + (0, −1, 1, 2) + (0, 2, −2, 6) + (6, 8, 4, −8) = (0, 0, 0, 0).

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(c) By Result 1 (its proof), a vector is a combination of the others exactly when some relation gives it a nonzero weight. The relation $-2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$ has nonzero weights on $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $-\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_4 + \mathbf{v}_5 = \mathbf{0}$ on $\mathbf{v}_4, \mathbf{v}_5$. So every one of the five is a combination of the other four; for example $\mathbf{v}_4 = \mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_5$.

4. Show by row reduction that $\mathbf{v}_1 = (1, -2, -3, 1, -3)$, $\mathbf{v}_2 = (1, 0, 1, 3, 3)$ and $\mathbf{v}_3 = (6, -4, -2, 14, 6)$ in $\mathbb{R}^5$ are linearly dependent. Find every dependence relation among them, and use one to write $\mathbf{v}_1$ as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$.

ANSWER

The dependence relations are $c_1 = -2t$, $c_2 = -4t$, $c_3 = t$ for any $t \neq 0$. With $t = -1$, $2\mathbf{v}_1 + 4\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$, so $\mathbf{v}_1 = \dfrac{1}{2}\,(\mathbf{v}_3 - 4\mathbf{v}_2)$.

SOLUTION

By Result 3, row reduce the matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Clear the first column with $R_i - (\text{entry})R_1$ for $i = 2, \ldots, 5$:

$$\begin{bmatrix} 1 & 1 & 6 \\ -2 & 0 & -4 \\ -3 & 1 & -2 \\ 1 & 3 & 14 \\ -3 & 3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 6 \\ 0 & 2 & 8 \\ 0 & 4 & 16 \\ 0 & 2 & 8 \\ 0 & 6 & 24 \end{bmatrix}$$

In each of the last four rows the third entry is 4 times the second. So the operations that clear column 2 below its pivot 2 clear column 3 as well, and the echelon form has pivots in columns 1 and 2 only. Column 3 has no pivot, so the list is dependent.

Set $c_3 = t$ and back substitute. Row 2 says 2 times c_2 plus 8 times t is 0, so $c_2 = -4t$. Row 1 then gives $c_1 = -(1)c_2 - (6)t = -2t$. Every relation is

$$-2t\,\mathbf{v}_1 - 4t\,\mathbf{v}_2 + t\,\mathbf{v}_3 = \mathbf{0}, \qquad t \neq 0.$$

With $t = -1$: $2\mathbf{v}_1 + 4\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$. The weight on $\mathbf{v}_1$ is not zero, so solve for it: $2\mathbf{v}_1 = \mathbf{v}_3 - 4\mathbf{v}_2$, and

$$\mathbf{v}_1 = \frac{1}{2}\,(\mathbf{v}_3 - 4\mathbf{v}_2)\,.$$

Check the first entry: 2 · 1 + 4 · 1 = 6.

5. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be a linearly independent list in $\mathbb{R}^n$, and let $\mathbf{w}$ be a vector of $\mathbb{R}^n$ that is not a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

(a) Prove that the list $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}$ is linearly independent.

(b) Give actual vectors in $\mathbb{R}^2$ showing that the conclusion can fail if $\mathbf{w}$ is a linear combination of the others.

ANSWER

(a) In any relation $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + d\mathbf{w} = \mathbf{0}$, $d \neq 0$ would make $\mathbf{w}$ a combination of the $\mathbf{v}_i$, so $d = 0$; then independence of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ forces every $c_i = 0$. (b) For example $\mathbf{v}_1 = (1, 0)$ and $\mathbf{w} = (2, 0)$: $\mathbf{v}_1$ alone is independent, but $2\mathbf{v}_1 - \mathbf{w} = \mathbf{0}$.

SOLUTION

(a) Suppose $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + d\mathbf{w} = \mathbf{0}$. We show every weight is 0.

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First, $d = 0$. If $d \neq 0$, move the other terms across and divide by d:

$$\mathbf{w} = -\frac{c_1}{d}\mathbf{v}_1 - \cdots - \frac{c_k}{d}\mathbf{v}_k,$$

which makes $\mathbf{w}$ a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, against the hypothesis.

With $d = 0$ the equation reads $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$. The list $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is independent, so $c_1 = \cdots = c_k = 0$. Only the trivial combination gives $\mathbf{0}$, so $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}$ is independent.

(b) Take $k = 1$, $\mathbf{v}_1 = (1, 0)$ and $\mathbf{w} = (2, 0) = 2\mathbf{v}_1$. The list $\mathbf{v}_1$ is independent because $\mathbf{v}_1 \neq \mathbf{0}$, but $2\mathbf{v}_1 - \mathbf{w} = \mathbf{0}$ is a dependence relation, so $\mathbf{v}_1, \mathbf{w}$ is dependent.

SCORING

Homework: (a) needs both steps, $d = 0$ (because otherwise $\mathbf{w}$ is a combination, with the division by d shown) and then every $c_i = 0$ by the independence of $\mathbf{v}_1, \ldots, \mathbf{v}_k$; (b) needs specific vectors and an explicit dependence relation.

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