Linear Algebra Unit 1 worksheet: Span (answer key)

ANSWERS AT A GLANCE

1. (a) all of $\mathbb{R}^2$ (b) a line through the origin (c) a plane through the origin (d) all of $\mathbb{R}^3$ (e) a line through the origin (f) a line through the origin
2. (a) $-x - 2y + z = 0$ (or any nonzero multiple of it). (b) $\mathbf{b}$ is in the span: $\mathbf{b} = 3\mathbf{v}_1 + 3\mathbf{v}_2$. $\mathbf{d}$ is not in the span.
3. (a) A line for $h = 2$ and $h = -2$; a plane for every other h. (b) The plane $2x - z = 0$ for every $h \neq \pm 2$. (c) $h = 2$: $2x - y = 0$ and $2x - z = 0$. $h = -2$: $2x + y = 0$ and $2x - z = 0$.
4. (a) No. (b) $x_1 - x_2 + 2x_3 = 0$ (or any nonzero multiple). (c) Only $\mathbf{e}_4$. (d) $j = 1$, 2 or 3; not $j = 4$.
5. (a) True. (b) True. (c) False: for example $\mathbf{v}_1 = \mathbf{e}_1$, $\mathbf{v}_2 = \mathbf{e}_2$, $\mathbf{v}_3 = \mathbf{e}_3$ and $\mathbf{b} = (0, 1, 1)$.
6. (a) $x - y + z = 0$. (b) (3, 1, −2): yes, $c_1 = \frac{5}{3}$ and $c_2 = -\frac{1}{3}$. (1, 1, 1): no. (c) Exactly the changes $(t, t, 0)$ for real t, rooms 1 and 2 changing by the same amount; both dampers move $\frac{t}{3}$ notches.

WORKED SOLUTIONS

1. Describe each span as the origin alone, a line through the origin, a plane through the origin, or all of $\mathbb{R}^2$ or $\mathbb{R}^3$. Give a reason for each.

(a) $\text{span}\{(2, 5), (-1, 3)\}$

(b) $\text{span}\{(-2, 4, -6), (3, -6, 9)\}$

(c) $\text{span}\{(1, 0, 2), (0, 3, 0), (2, 3, 4)\}$

(d) $\text{span}\{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$

(e) $\text{span}\{(3, -1, 2, 0)\}$

(f) $\text{span}\{(1, 2), (2, 4), (-3, -6)\}$

ANSWER

(a) all of $\mathbb{R}^2$ (b) a line through the origin (c) a plane through the origin (d) all of $\mathbb{R}^3$ (e) a line through the origin (f) a line through the origin

SOLUTION

(a) Neither vector is a multiple of the other: a multiple of (2, 5) with first entry −1 is $-\frac{1}{2}(2, 5) = \left(-1, -\frac{5}{2}\right) \neq (-1, 3)$. Two such vectors in $\mathbb{R}^2$ span all of $\mathbb{R}^2$.

(b) (−2, 4, −6) = −2(1, −2, 3) and (3, −6, 9) = 3(1, −2, 3), so each is a multiple of the other and the span is $\text{span}\{(1, -2, 3)\}$: the line through the origin along (1, −2, 3).

(c) (2, 3, 4) = 2(1, 0, 2) + (0, 3, 0) is already in the span of the first two, so it adds nothing. Neither of (1, 0, 2) and (0, 3, 0) is a multiple of the other, so the span is the plane through the origin containing them.

(d) With the vectors as columns, $R_2 - R_1$ and then $R_3 - R_2$ give

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix},$$

a pivot in every row, so the vectors span all of $\mathbb{R}^3$.

(e) One nonzero vector spans the line through the origin along it, here a line in $\mathbb{R}^4$.

(f) (2, 4) = 2(1, 2) and (−3, −6) = −3(1, 2), so the span is $\text{span}\{(1, 2)\}$, the line $y = 2x$ through the origin.

2. Let $\mathbf{v}_1 = (1, -2, -3)$ and $\mathbf{v}_2 = (-2, 5, 8)$ in $\mathbb{R}^3$.

(a) Row reduce with a general right side (x, y, z) to find one equation in x, y and z satisfied by exactly the vectors of $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

(b) Use your equation to decide whether $\mathbf{b} = (-3, 9, 15)$ and $\mathbf{d} = (-1, 9, 15)$ are in the span. For each one that is, find the weights.

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ANSWER

(a) $-x-2y+z=0$ (or any nonzero multiple of it). (b) $\mathbf{b}$ is in the span: $\mathbf{b}=3\mathbf{v}_1+3\mathbf{v}_2$. $\mathbf{d}$ is not in the span.

SOLUTION

(a) Reduce the augmented matrix with columns $\mathbf{v}_1$, $\mathbf{v}_2$ and a general right side (x,y,z). First R_2+2R_1 and R_3+3R_1, then R_3-2R_2:

$$\left[\begin{array}{cc|c} 1 & -2 & x \\ -2 & 5 & y \\ -3 & 8 & z \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & x \\ 0 & 1 & y+2x \\ 0 & 2 & z+3x \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & x \\ 0 & 1 & y+2x \\ 0 & 0 & -x-2y+z \end{array}\right]$$

The last entry is $z+3x-2\,(y+2x)=-x-2y+z$. When it is 0, rows 1 and 2 give weights; when it is not, the last row reads 0 = (nonzero). So the span is the plane

$$-x-2y+z=0.$$

(b) For $\mathbf{b}$: $-1\cdot(-3)-2\cdot 9+15=0$, so $\mathbf{b}$ is in the span. Row 2 gives $c_2=9+2\cdot(-3)=3$ and row 1 gives $c_1=-3+2\cdot 3=3$, so $\mathbf{b}=3\mathbf{v}_1+3\mathbf{v}_2$.

For $\mathbf{d}$: $-1\cdot(-1)-2\cdot 9+15=-2\neq 0$, so $\mathbf{d}$ is not in the span.

3. For a number h, let $\mathbf{v}_1=(1,h,2)$ and $\mathbf{v}_2=(h,4,2h)$ in $\mathbb{R}^3$.

(a) For which values of h is $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2\}$ a line? What is it for every other h?

(b) Show that for every h not found in (a) the span is one and the same set, and give its equation.

(c) For each h found in (a), give two equations describing the line.

ANSWER

(a) A line for $h=2$ and $h=-2$; a plane for every other h. (b) The plane $2x-z=0$ for every $h\neq\pm 2$. (c) $h=2$: $2x-y=0$ and $2x-z=0$. $h=-2$: $2x+y=0$ and $2x-z=0$.

SOLUTION

Reduce $[\,\mathbf{v}_1\ \mathbf{v}_2\mid\mathbf{b}\,]$ with $\mathbf{b}=(x,y,z)$, using R_2-hR_1 and R_3-2R_1:

$$\left[\begin{array}{cc|c} 1 & h & x \\ h & 4 & y \\ 2 & 2h & z \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & h & x \\ 0 & 4-h^2 & y-hx \\ 0 & 0 & z-2x \end{array}\right]$$

(a) Column 2 has a pivot exactly when $4-h^2\neq 0$. When $h=\pm 2$ it has none and $\mathbf{v}_2=h\mathbf{v}_1$: (2, 4, 4) = 2(1, 2, 2) and (−2, 4, −4) = −2(1, −2, 2). Then the span is $\operatorname{span}\{\mathbf{v}_1\}$, a line. For every other h neither vector is a multiple of the other ($\mathbf{v}_1\neq\mathbf{0}$, and $\mathbf{v}_2=t\mathbf{v}_1$ would need $t=h$ and $4=h^2$), so the span is a plane through the origin.

(b) For $h\neq\pm 2$, row 2 has a pivot, so the system is consistent exactly when the row 3 entry is 0: $z-2x=0$. That condition does not involve h, so every such h gives the same plane

$$2x-z=0.$$

Check: $\mathbf{v}_1$ gives 2 − 2 = 0 and $\mathbf{v}_2$ gives $2h-2h=0$.

(c) For $h=\pm 2$, rows 2 and 3 are both zero on the left, so both right sides must be 0: $y-hx=0$ and $z-2x=0$. For $h=2$: $2x-y=0$ and $2x-z=0$. For $h=-2$: $2x+y=0$ and $2x-z=0$.

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4. Let $\mathbf{v}_1 = (1, 1, 0, 1)$, $\mathbf{v}_2 = (-2, 0, 1, 2)$, $\mathbf{v}_3 = (0, 2, 1, -1)$ and $\mathbf{v}_4 = (3, 1, -1, 2)$ in $\mathbb{R}^4$, and let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ be the standard vectors of $\mathbb{R}^4$.

(a) Do $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span $\mathbb{R}^4$?

(b) Find one equation in x_1, x_2, x_3, x_4 satisfied by exactly the vectors of $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

(c) Which of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ are in this span?

(d) For which j do the five vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{e}_j$ span $\mathbb{R}^4$? Justify with the pivot test.

ANSWER

(a) No. (b) $x_1 - x_2 + 2x_3 = 0$ (or any nonzero multiple). (c) Only $\mathbf{e}_4$. (d) $j = 1$, 2 or 3; not $j = 4$.

SOLUTION

(a) and (b) Reduce $[\,\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4 \mid \mathbf{x}\,]$ with a general $\mathbf{x} = (x_1, x_2, x_3, x_4)$. First $R_2 - R_1$ and $R_4 - R_1$, then swap rows 2 and 3:

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & x_1 \\ 1 & 0 & 2 & 1 & x_2 \\ 0 & 1 & 1 & -1 & x_3 \\ 1 & 2 & -1 & 2 & x_4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & x_1 \\ 0 & 1 & 1 & -1 & x_3 \\ 0 & 2 & 2 & -2 & x_2 - x_1 \\ 0 & 4 & -1 & -1 & x_4 - x_1 \end{array}\right]$$

Then $R_3 - 2R_2$ and $R_4 - 4R_2$, and swap rows 3 and 4:

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & x_1 \\ 0 & 1 & 1 & -1 & x_3 \\ 0 & 0 & -5 & 3 & x_4 - x_1 - 4x_3 \\ 0 & 0 & 0 & 0 & -x_1 + x_2 - 2x_3 \end{array}\right]$$

The echelon form of the coefficient part has no pivot in row 4, so the four vectors do not span $\mathbb{R}^4$. The system is consistent exactly when the last entry is 0, so the span is

$$x_1 - x_2 + 2x_3 = 0.$$

Check: $\mathbf{v}_1$ gives $1 - 1 + 0 = 0$, $\mathbf{v}_2$ gives $-2 - 0 + 2 = 0$, $\mathbf{v}_3$ gives $0 - 2 + 2 = 0$ and $\mathbf{v}_4$ gives $3 - 1 - 2 = 0$.

(c) Put each $\mathbf{e}_j$ into the equation: $\mathbf{e}_1$ gives 1, $\mathbf{e}_2$ gives -1, $\mathbf{e}_3$ gives 2 and $\mathbf{e}_4$ gives 0. Only $\mathbf{e}_4$ is in the span.

(d) Add $\mathbf{e}_j$ as a fifth column. The same row operations act on it, so its entry in row 4 becomes $-x_1 + x_2 - 2x_3$ evaluated at $\mathbf{e}_j$: -1, 1, -2 or 0 for $j = 1, 2, 3, 4$. For $j = 1, 2, 3$ that entry is nonzero and is a pivot in row 4, so the echelon form of $[\,\mathbf{v}_1\ \cdots\ \mathbf{v}_4\ \mathbf{e}_j\,]$ has a pivot in every row and the five vectors span $\mathbb{R}^4$. For $j = 4$ row 4 stays all zeros: $\mathbf{e}_4$ is already in the span and adds nothing, so the five vectors do not span $\mathbb{R}^4$.

5. Decide whether each statement about vectors in $\mathbb{R}^3$ is true or false. Give a reason for a true statement and a counterexample with actual vectors for a false one.

(a) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$, then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span $\mathbb{R}^3$ for every $\mathbf{v}_4$.

(b) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$, then $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane.

(c) If $\mathbf{b}$ is not in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and not in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_3\}$, then $\mathbf{b}$ is not in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

ANSWER

(a) True. (b) True. (c) False: for example $\mathbf{v}_1 = \mathbf{e}_1$, $\mathbf{v}_2 = \mathbf{e}_2$, $\mathbf{v}_3 = \mathbf{e}_3$ and $\mathbf{b} = (0, 1, 1)$.

SOLUTION

(a) True. Any $\mathbf{b}$ in $\mathbb{R}^3$ is $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ for some weights, and then $\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + 0\mathbf{v}_4$. So every $\mathbf{b}$ is in the span of the four.

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(b) True. If $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is not a plane, one of the two is a multiple of the other, so the span is $\operatorname{span}\{\mathbf{u}\}$ with $\mathbf{u}$ one of them, and the other one is in $\operatorname{span}\{\mathbf{u}\}$. By the adding result $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \operatorname{span}\{\mathbf{u}, \mathbf{v}_3\}$. Two vectors never span $\mathbb{R}^3$ (an echelon form of a 3×2 matrix has at most two pivots), which contradicts the hypothesis.

(c) False. Take $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (0, 1, 0)$, $\mathbf{v}_3 = (0, 0, 1)$ and $\mathbf{b} = (0, 1, 1)$. Every vector of $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is $(c_1, c_2, 0)$, with third entry 0, so $\mathbf{b}$ is not in it. Every vector of $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_3\}$ is $(c_1, 0, c_3)$, with second entry 0, so $\mathbf{b}$ is not in it either. But $\mathbf{b} = \mathbf{v}_2 + \mathbf{v}_3$ is in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

6. Priya's office has three rooms on one air system with two dampers. Moving damper A by one notch from its usual setting changes the temperatures of (room 1, room 2, room 3) by $\mathbf{a} = (2, 1, -1)$ degrees Celsius; moving damper B by one notch changes them by $\mathbf{b} = (1, 2, 1)$. In this made-up model a damper can move any real number of notches (a negative number moves it the other way), and the changes add: settings c_1 and c_2 change the temperatures by $c_1\mathbf{a} + c_2\mathbf{b}$.

(a) Find one equation satisfied by exactly the changes (x, y, z) the two dampers can make.

(b) Can the dampers change the temperatures by (3, 1, −2)? By (1, 1, 1)? Where they can, give the settings.

(c) Describe every change the dampers can make that leaves room 3 unchanged, and the settings that make it.

ANSWER

(a) $x - y + z = 0$. (b) (3, 1, −2): yes, $c_1 = \dfrac{5}{3}$ and $c_2 = -\dfrac{1}{3}$. (1, 1, 1): no. (c) Exactly the changes $(t, t, 0)$ for real t, rooms 1 and 2 changing by the same amount; both dampers move $\dfrac{t}{3}$ notches.

SOLUTION

(a) The possible changes are $\operatorname{span}\{\mathbf{a}, \mathbf{b}\}$. Reduce with a general right side: swap rows 1 and 2, then $R_2 - 2R_1$ and $R_3 + R_1$, then $R_3 + R_2$:

$$\left[\begin{array}{cc|c} 2 & 1 & x \\ 1 & 2 & y \\ -1 & 1 & z \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & y \\ 0 & -3 & x - 2y \\ 0 & 3 & y + z \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & y \\ 0 & -3 & x - 2y \\ 0 & 0 & x - y + z \end{array}\right]$$

The system is consistent exactly when $x - y + z = 0$. Check: $\mathbf{a}$ gives 2 − 1 − 1 = 0 and $\mathbf{b}$ gives 1 − 2 + 1 = 0.

(b) (3, 1, −2): 3 − 1 − 2 = 0, so yes. Row 2 gives $-3c_2 = 3 - 2 = 1$, so $c_2 = -\dfrac{1}{3}$; row 1 gives $c_1 + 2c_2 = 1$, so $c_1 = \dfrac{5}{3}$. Check: $\left(\dfrac{10}{3}, \dfrac{5}{3}, -\dfrac{5}{3}\right) + \left(-\dfrac{1}{3}, -\dfrac{2}{3}, -\dfrac{1}{3}\right) = (3, 1, -2)$.

(1, 1, 1): 1 − 1 + 1 = 1 ≠ 0, so no settings make it.

(c) Room 3 unchanged means $z = 0$, and then the equation gives $x = y$. So the changes are exactly $(t, t, 0)$ for real t: a line through the origin. The third entry of $c_1\mathbf{a} + c_2\mathbf{b}$ is $-c_1 + c_2$, which is 0 exactly when $c_1 = c_2$, and then $c_1(\mathbf{a} + \mathbf{b}) = (3c_1, 3c_1, 0)$. So both dampers move the same amount, $\dfrac{t}{3}$ notches.

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