Linear Algebra Unit 1 worksheet: Subspaces (answer key)

ANSWERS AT A GLANCE

1. (a) Yes. (b) No: (0, 1) is in it but (−1)(0, 1) = (0, −1) is not. (c) No: (1, 1, 1) is in it but 2(1, 1, 1) = (2, 2, 2) is not. (d) No: (1, 0, 0) and (0, 1, 0) are in it but their sum (1, 1, 0) is not. (e) No: it does not contain **0**. (f) Yes: it is $\{\mathbf{0}\}$.
2. (a) W contains **0** and is closed under addition and under scaling by every real c, so it is a subspace. (b) $W = \text{span}\{(1,0,0,1),(0,1,1,0)\}$ (any list with the same span is accepted). (c) No: the zero table has all its sums 0, not 10.
3. $N = \text{span}\{(6,-2,1,0,0),(3,-2,0,-2,1)\}$ (any list with the same span is accepted). N is a subspace because it is the solution set of a homogeneous system.
4. (a) V is a span and H is the solution set of a homogeneous equation, so both are subspaces. (b) $V \cap H = \text{span}\{(2,-1,3,3)\}$, a line through the origin (any nonzero multiple of the vector is accepted); every span is a subspace. (c) No: (1, 0, 1, 2) is in V but 1 + 0 − 1 + 2 = 2 ≠ 0, so it is not in H.
5. (a) $\mathbf{0} \notin W_1$: the last two entries force $s = -1$, $t = 0$, and then the first entry is −1, not 0. (b) $W_2 = \text{span}\{(2,1,3,0),(-1,1,0,-1)\}$ (any list with the same span is accepted). (c) Yes: $s = 2$, $t = 2$ give it, and (5, 4, 9, −1) = 3(2, 1, 3, 0) + (−1, 1, 0, −1).
6. See the proof below.

WORKED SOLUTIONS

1. Decide whether each set is a subspace. For a yes, give the reason; for a no, show one condition failing with specific vectors.

(a) $\{(x,y,z) : x = 3z\}$ in $\mathbb{R}^3$

(b) $\{(x,y) : x \le y\}$ in $\mathbb{R}^2$

(c) $\{(x,y,z) : z = xy\}$ in $\mathbb{R}^3$

(d) $\{(x,y,z) : x = 0 \text{ or } y = 0\}$ in $\mathbb{R}^3$

(e) $\{(x,y,z) : 2x - y = 0 \text{ and } z = 5\}$ in $\mathbb{R}^3$

(f) $\{(x,y,z) : x^2 + y^2 + z^2 = 0\}$ in $\mathbb{R}^3$

ANSWER

(a) Yes. (b) No: (0, 1) is in it but (−1)(0, 1) = (0, −1) is not. (c) No: (1, 1, 1) is in it but 2(1, 1, 1) = (2, 2, 2) is not. (d) No: (1, 0, 0) and (0, 1, 0) are in it but their sum (1, 1, 0) is not. (e) No: it does not contain **0**. (f) Yes: it is $\{\mathbf{0}\}$.

SOLUTION

(a) It is the solution set of the homogeneous equation $x - 3z = 0$, so it is a subspace. (Solving $x = 3z$ with y, z free, it is $\text{span}\{(3,0,1),(0,1,0)\}$.)

(b) It contains **0** and is closed under addition (add the two inequalities), but not under scaling: (0, 1) has 0 ≤ 1, while (−1)(0, 1) = (0, −1) has 0 > −1. Not a subspace.

(c) (1, 1, 1) is in the set, since 1 = 1 · 1. But 2(1, 1, 1) = (2, 2, 2) is not, since 2 ≠ 2 · 2. Not a subspace.

(d) (1, 0, 0) has $y = 0$ and (0, 1, 0) has $x = 0$, so both are in the set, but (1, 0, 0) + (0, 1, 0) = (1, 1, 0) has $x \neq 0$ and $y \neq 0$. Not a subspace.

(e) The zero vector has third entry 0 ≠ 5, so **0** is not in the set. Not a subspace.

(f) A sum of squares of real numbers is 0 only when each number is 0, so the set is $\{(0,0,0)\}$, the zero subspace.

2. A 2 × 2 table of numbers with rows (p, q) and (r, s) is recorded as the vector (p, q, r, s) in $\mathbb{R}^4$. Call a table balanced when its two row sums are equal and its two column sums are equal.

(a) Show from the three conditions, using general vectors, that the balanced tables form a subspace W of $\mathbb{R}^4$.

(b) Write W as a span.

(c) Do the tables whose two row sums and two column sums all equal 10 form a subspace? Explain.

ANSWER

(a) W contains **0** and is closed under addition and under scaling by every real c, so it is a subspace. (b) $W = \text{span}\{(1,0,0,1),(0,1,1,0)\}$ (any list with the same span is accepted). (c) No: the zero table has all

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its sums 0, not 10.

SOLUTION

A table (p, q, r, s) is balanced when $p + q = r + s$ (rows) and $p + r = q + s$ (columns).

(a) *Zero*. The zero table has 0 + 0 = 0 + 0 for both, so $\mathbf{0} \in W$.

Addition. Let $\mathbf{u} = (p, q, r, s)$ and $\mathbf{v} = (p', q', r', s')$ be balanced. Regrouping,

$$(p + p') + (q + q') = (p + q) + (p' + q') = (r + s) + (r' + s') = (r + r') + (s + s'),$$
$$(p + p') + (r + r') = (p + r) + (p' + r') = (q + s) + (q' + s') = (q + q') + (s + s'),$$

so $\mathbf{u} + \mathbf{v}$ is balanced.

Scaling. For any real c, $cp + cq = c(p + q) = c(r + s) = cr + cs$, and likewise $cp + cr = c(q + s) = cq + cs$, so $c\mathbf{u}$ is balanced.

All three hold, so W is a subspace.

(b) Subtracting the column equation from the row equation gives $q - r = r - q$, so $q = r$; adding them gives $2p + q + r = q + r + 2s$, so $p = s$. Conversely every (p, q, q, p) is balanced. So

$$W = \{(p, q, q, p)\} = \{p(1, 0, 0, 1) + q(0, 1, 1, 0)\} = \operatorname{span}\{(1, 0, 0, 1), (0, 1, 1, 0)\}.$$

(c) No. The zero table is not in this set, since its sums are 0, not 10. (The set is not empty: the table with every entry 5 is in it.)

3. Let

$$A = \begin{bmatrix} 1 & 2 & -2 & -2 & -3 \\ -1 & -1 & 4 & 2 & 5 \\ -2 & -3 & 6 & 5 & 10 \end{bmatrix}.$$

Write the solution set N of $A\mathbf{x} = \mathbf{0}$ as a span, and say why N is a subspace of $\mathbb{R}^5$.

ANSWER

$N = \operatorname{span}\{(6, -2, 1, 0, 0), (3, -2, 0, -2, 1)\}$ (any list with the same span is accepted). N is a subspace because it is the solution set of a homogeneous system.

SOLUTION

N is the solution set of a homogeneous system, so it is a subspace of $\mathbb{R}^5$. Row reduce: add 1 times row 1 to row 2 and 2 times row 1 to row 3, then add −1 times row 2 to row 3:

$$A \sim \begin{bmatrix} 1 & 2 & -2 & -2 & -3 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & -2 & -3 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

The pivots are in columns 1, 2 and 4, so x_3 and x_5 are free. Row 3 gives x_4, row 2 gives x_2 and row 1 then gives x_1, each in terms of x_3 and x_5:

$$\mathbf{x} = x_3 \begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

So $N = \operatorname{span}\{(6, -2, 1, 0, 0), (3, -2, 0, -2, 1)\}$. Check: A times each spanning vector is (0, 0, 0).

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4. In $\mathbb{R}^4$, let $V = \text{span}\{(1,0,1,2), (0,1,-1,1)\}$ and let H be the set of vectors with $x_1 + 2x_2 - x_3 + x_4 = 0$.

(a) Explain why V and H are subspaces.

(b) Write the intersection $V \cap H$ (the vectors in both) as a span, and say why this shows $V \cap H$ is a subspace.

(c) Is V contained in H? Explain.

ANSWER

(a) V is a span and H is the solution set of a homogeneous equation, so both are subspaces. (b) $V \cap H = \text{span}\{(2,-1,3,3)\}$, a line through the origin (any nonzero multiple of the vector is accepted); every span is a subspace. (c) No: (1, 0, 1, 2) is in V but 1 + 0 − 1 + 2 = 2 ≠ 0, so it is not in H.

SOLUTION

(a) Every span is a subspace, and the solution set of a homogeneous equation is a subspace.

(b) Every vector of V is

$$a(1,0,1,2) + b(0,1,-1,1) = (a,\ b,\ a-b,\ 2a+b).$$

It lies in H exactly when $a + 2b - (a - b) + (2a + b) = 2a + 4b = 0$, that is, $a = -2b$. With $b = -1$ and $a = 2$ this gives (2, −1, 3, 3), and every vector of $V \cap H$ is $-b(2,-1,3,3)$. So $V \cap H = \text{span}\{(2,-1,3,3)\}$, a line through the origin, and it is a subspace because every span is. Check: 2 − 2 − 3 + 3 = 0.

(c) No. $(1,0,1,2) \in V$, but 1 + 2(0) − 1 + 2 = 2 ≠ 0, so it is not in H. (This is also why $V \cap H$ is smaller than V.)

5. For real numbers s and t, let W_1 be the set of all vectors $(2s - t + 1,\ s + t + 2,\ 3s + 3,\ -t)$ and W_2 the set of all vectors $(2s - t + 3,\ s + t,\ 3s + 3,\ 1 - t)$ in $\mathbb{R}^4$. Each looks like a plane slid off the origin.

(a) Show that W_1 is not a subspace.

(b) Show that W_2 is a subspace by writing it as a span. (Hint: first find the s and t that give $\mathbf{0}$.)

(c) Decide whether (5, 4, 9, −1) is in W_2.

ANSWER

(a) $\mathbf{0} \notin W_1$: the last two entries force $s = -1$, $t = 0$, and then the first entry is −1, not 0. (b) $W_2 = \text{span}\{(2,1,3,0), (-1,1,0,-1)\}$ (any list with the same span is accepted). (c) Yes: $s = 2$, $t = 2$ give it, and (5, 4, 9, −1) = 3(2, 1, 3, 0) + (−1, 1, 0, −1).

SOLUTION

(a) For $(2s - t + 1, s + t + 2, 3s + 3, -t)$ to be $\mathbf{0}$, the fourth entry needs $t = 0$ and the third needs $s = -1$. Then the first entry is −2 − 0 + 1 = −1 ≠ 0. So $\mathbf{0} \notin W_1$, and W_1 is not a subspace.

(b) $s = -1, t = 1$ give $(-2 - 1 + 3,\ -1 + 1,\ -3 + 3,\ 1 - 1) = \mathbf{0}$, so measure from there: let $S = s + 1$ and $T = t - 1$, so $s = S - 1$ and $t = T + 1$. Then

$$2s - t + 3 = 2S - T, \quad s + t = S + T, \quad 3s + 3 = 3S, \quad 1 - t = -T,$$

and as s and t run over all real numbers, so do S and T. So W_2 is the set of all

$$(2S - T,\ S + T,\ 3S,\ -T) = S(2,1,3,0) + T(-1,1,0,-1),$$

that is, $W_2 = \text{span}\{(2,1,3,0), (-1,1,0,-1)\}$. Every span is a subspace.

(c) The third and fourth entries of (5, 4, 9, −1) force $3s + 3 = 9$ and $1 - t = -1$, so $s = 2$ and $t = 2$. These give (4 − 2 + 3, 2 + 2, 9, −1) = (5, 4, 9, −1), so it is in W_2. In the span, $S = 3$ and $T = 1$: 3(2, 1, 3, 0) + (−1, 1, 0, −1) = (5, 4, 9, −1).

6. Let U and W be subspaces of $\mathbb{R}^n$, and let $U + W$ be the set of all sums $\mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U$ and $\mathbf{w} \in W$.

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(a) Prove that $U + W$ is a subspace of $\mathbb{R}^n$.

(b) In $\mathbb{R}^3$, let $U = \text{span}\{(1, 0, 0)\}$ and $W = \text{span}\{(0, 1, 1)\}$. Write $U + W$ as a span and give an equation for it.

ANSWER

(a) $\mathbf{0} = \mathbf{0} + \mathbf{0} \in U + W$, and $c(\mathbf{u}_1 + \mathbf{w}_1) + d(\mathbf{u}_2 + \mathbf{w}_2) = (c\mathbf{u}_1 + d\mathbf{u}_2) + (c\mathbf{w}_1 + d\mathbf{w}_2)$ is a vector of U plus a vector of W. (b) $U + W = \text{span}\{(1, 0, 0), (0, 1, 1)\}$, the plane $y - z = 0$.

SOLUTION

(a) *Zero.* $\mathbf{0} \in U$ and $\mathbf{0} \in W$, so $\mathbf{0} = \mathbf{0} + \mathbf{0} \in U + W$.

One-line test. Let $\mathbf{x} = \mathbf{u}_1 + \mathbf{w}_1$ and $\mathbf{y} = \mathbf{u}_2 + \mathbf{w}_2$ be in $U + W$, with $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{w}_1, \mathbf{w}_2 \in W$, and let c, d be any scalars. Distributing and regrouping (the properties of vector addition),

$$c\mathbf{x} + d\mathbf{y} = (c\mathbf{u}_1 + d\mathbf{u}_2) + (c\mathbf{w}_1 + d\mathbf{w}_2).$$

U is a subspace, so $c\mathbf{u}_1 + d\mathbf{u}_2 \in U$; W is a subspace, so $c\mathbf{w}_1 + d\mathbf{w}_2 \in W$. So $c\mathbf{x} + d\mathbf{y}$ is a vector of U plus a vector of W, and lies in $U + W$. By the one-line test, $U + W$ is a subspace.

(b) A vector of U is $a(1, 0, 0)$ and a vector of W is $b(0, 1, 1)$, so $U + W$ is the set of all $a(1, 0, 0) + b(0, 1, 1) = (a, b, b)$: that is, $\text{span}\{(1, 0, 0), (0, 1, 1)\}$. A vector (x, y, z) has this form exactly when $y = z$, so $U + W$ is the plane $y - z = 0$.

SCORING

Homework: (a) needs $\mathbf{0}$ in $U + W$, general vectors of $U + W$ written as sums, and the step that uses each of U and W being a subspace; (b) needs the span and the equation.

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