Linear Algebra Unit 1 worksheet: Adding and scaling vectors in n dimensions (answer key)

ANSWERS AT A GLANCE

1. (a) $(2, 5, -4, 3, 5, 8)$ (b) $(-7, 12, 17, -12, -21, 11)$ (c) $(6, -7, -6, 9, 5, 0)$ (d) $(-\frac{1}{2}, \frac{5}{2}, \frac{11}{2}, -\frac{3}{2}, -8, \frac{11}{2})$
2. $\mathbf{x} = (2, 5, 1, -3)$
3. (a) $x = -2$, $y = 3$ only. (b) $t = -1$ only. (c) $k = 2$ only.
4. $\mathbf{x} = (4, -4, -1, -4)$ and $\mathbf{y} = (3, 3, 3, 2)$
5. (a) $\mathbf{x} = -\frac{5}{2}\mathbf{u} - \mathbf{v}$ (b) $\mathbf{x} = (-6, 7, 2, -10)$; both sides equal $(-28, 51, 2, -76)$.
6. (a) $\mathbf{m} = (1, 5, 1, 3)$ (b) $\mathbf{g} = (3, 2, 1, 0)$ (c) $\frac{1}{3}(\mathbf{p} + \mathbf{q} + \mathbf{r}) = \frac{1}{3}(9, 6, 3, 0) = (3, 2, 1, 0)$, matching (b). (d) The expression $\frac{1}{3}(\mathbf{p} + \mathbf{q} + \mathbf{r})$ does not change when the three points swap roles.
7. See the proof below.

WORKED SOLUTIONS

1. Let $\mathbf{u} = (3, -1, 0, 4, -2, 5)$, $\mathbf{v} = (-2, 6, 1, -3, 0, 4)$ and $\mathbf{w} = (1, 0, -5, 2, 7, -1)$ in $\mathbb{R}^6$. Compute each vector.

(a) $\mathbf{u} + \mathbf{v} + \mathbf{w}$

(b) $2\mathbf{v} - 3\mathbf{w}$

(c) $\mathbf{u} - (\mathbf{v} - \mathbf{w})$

(d) $\frac{1}{2}(\mathbf{u} + \mathbf{v}) - \mathbf{w}$

ANSWER

(a) $(2, 5, -4, 3, 5, 8)$ (b) $(-7, 12, 17, -12, -21, 11)$ (c) $(6, -7, -6, 9, 5, 0)$ (d) $(-\frac{1}{2}, \frac{5}{2}, \frac{11}{2}, -\frac{3}{2}, -8, \frac{11}{2})$

SOLUTION

(a) Add position by position: $(3 - 2 + 1, -1 + 6 + 0, 0 + 1 - 5, 4 - 3 + 2, -2 + 0 + 7, 5 + 4 - 1) = (2, 5, -4, 3, 5, 8)$.

(b) $2\mathbf{v} = (-4, 12, 2, -6, 0, 8)$ and $3\mathbf{w} = (3, 0, -15, 6, 21, -3)$, so $2\mathbf{v} - 3\mathbf{w} = (-7, 12, 17, -12, -21, 11)$.

(c) $\mathbf{v} - \mathbf{w} = (-3, 6, 6, -5, -7, 5)$, so $\mathbf{u} - (\mathbf{v} - \mathbf{w}) = (6, -7, -6, 9, 5, 0)$. (The same as $\mathbf{u} - \mathbf{v} + \mathbf{w}$.)

(d) $\mathbf{u} + \mathbf{v} = (1, 5, 1, 1, -2, 9)$, so $\frac{1}{2}(\mathbf{u} + \mathbf{v}) = (\frac{1}{2}, \frac{5}{2}, \frac{1}{2}, \frac{1}{2}, -1, \frac{9}{2})$. Subtract $\mathbf{w}$: $(-\frac{1}{2}, \frac{5}{2}, \frac{11}{2}, -\frac{3}{2}, -8, \frac{11}{2})$.

2. Solve $3\mathbf{x} + \mathbf{a} = \mathbf{b} - 2\mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^4$, where $\mathbf{a} = (-1, -6, 7, 6)$ and $\mathbf{b} = (9, 19, 12, -9)$. Check your answer in the original equation.

ANSWER

$\mathbf{x} = (2, 5, 1, -3)$

SOLUTION

Add $2\mathbf{x}$ and $-\mathbf{a}$ to both sides. The properties let us regroup and cancel ($\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$), and property 6 combines $3\mathbf{x} + 2\mathbf{x} = 5\mathbf{x}$:

$$5\mathbf{x} = \mathbf{b} - \mathbf{a} = (10, 25, 5, -15).$$

Multiply both sides by $\frac{1}{5}$ (properties 7 and 8 turn $\frac{1}{5}(5\mathbf{x})$ into $\mathbf{x}$):

$$\mathbf{x} = (2, 5, 1, -3).$$

Check: $3\mathbf{x} + \mathbf{a}$ and $\mathbf{b} - 2\mathbf{x}$ both equal $(5, 9, 10, -3)$.

3. Use the definition of equal vectors.

(a) Find all real numbers x and y with $(x^2, x + y, y^2 - 1, xy) = (4, 1, 8, -6)$.

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(b) Find every real number t for which (t^2-1, t^2+t, t^3-t) is the zero vector of $\mathbb{R}^3$.

(c) Find every scalar k with $k(k, 1, 2, -k) = (4, 2, 4, -4)$.

ANSWER

(a) $x = -2$, $y = 3$ only. (b) $t = -1$ only. (c) $k = 2$ only.

SOLUTION

(a) Both sides are in $\mathbb{R}^4$, so there are four equations: $x^2 = 4$, $x + y = 1$, $y^2 - 1 = 8$ and $xy = -6$. The first gives $x = 2$ or $x = -2$, and the second then gives the pairs (2, −1) and (−2, 3). The third needs $y^2 = 9$, which rules out $y = -1$, leaving (−2, 3). The fourth holds: (−2)(3) = −6. So $x = -2$, $y = 3$ is the only solution.

(b) Every entry must be 0. $t^2 - 1 = 0$ gives $t = 1$ or $t = -1$. $t^2 + t = t(t+1) = 0$ gives $t = 0$ or $t = -1$. $t^3 - t = t(t-1)(t+1) = 0$ gives $t = 0$, 1 or −1. Only $t = -1$ is on all three lists.

(c) The left side is $(k^2, k, 2k, -k^2)$. The second position says $k = 2$. The others agree: $2^2 = 4$, $2 \cdot 2 = 4$ and $-2^2 = -4$. The first position alone would also allow $k = -2$, but the second rules it out, so $k = 2$ only.

4. Find the vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^4$ with $\mathbf{x} + 2\mathbf{y} = \mathbf{a}$ and $\mathbf{x} - \mathbf{y} = \mathbf{b}$, where $\mathbf{a} = (10, 2, 5, 0)$ and $\mathbf{b} = (1, -7, -4, -6)$.

ANSWER

$\mathbf{x} = (4, -4, -1, -4)$ and $\mathbf{y} = (3, 3, 3, 2)$

SOLUTION

Subtract the second equation from the first. The $\mathbf{x}$ terms cancel and property 6 combines $2\mathbf{y} + \mathbf{y} = 3\mathbf{y}$:

$$3\mathbf{y} = \mathbf{a} - \mathbf{b} = (9, 9, 9, 6),$$

so $\mathbf{y} = \dfrac{1}{3}(\mathbf{a} - \mathbf{b}) = (3, 3, 3, 2)$.

The second equation gives $\mathbf{x} = \mathbf{b} + \mathbf{y} = (4, -4, -1, -4)$.

Check in the first equation: $\mathbf{x} + 2\mathbf{y} = (10, 2, 5, 0) = \mathbf{a}$.

5. Vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{x}$ in $\mathbb{R}^n$ satisfy

$$3(\mathbf{x} - 2\mathbf{u}) + 2\mathbf{v} = 5\mathbf{x} - (\mathbf{u} - 4\mathbf{v}).$$

(a) Solve for $\mathbf{x}$ in terms of $\mathbf{u}$ and $\mathbf{v}$, naming the properties you use.

(b) Find $\mathbf{x}$ when $\mathbf{u} = (2, -4, 0, 6)$ and $\mathbf{v} = (1, 3, -2, -5)$, and check it in the original equation.

ANSWER

(a) $\mathbf{x} = -\dfrac{5}{2}\mathbf{u} - \mathbf{v}$ (b) $\mathbf{x} = (-6, 7, 2, -10)$; both sides equal (−28, 51, 2, −76).

SOLUTION

(a) Distribute with property 5 and combine scalars with property 7:

$$3\mathbf{x} - 6\mathbf{u} + 2\mathbf{v} = 5\mathbf{x} - \mathbf{u} + 4\mathbf{v}.$$

Add $-3\mathbf{x} + \mathbf{u} - 4\mathbf{v}$ to both sides. Reorder and regroup (properties 1 and 2), cancel each vector against its negative (properties 4 and 3), and combine the multiples of each vector (property 6):

$$-5\mathbf{u} - 2\mathbf{v} = 2\mathbf{x}.$$

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Multiply both sides by $\frac{1}{2}$ (property 5 on the left, properties 7 and 8 on the right):

$$\mathbf{x} = -\tfrac{5}{2}\mathbf{u} - \mathbf{v}.$$

(b) $-\frac{5}{2}\mathbf{u} = (-5, 10, 0, -15)$, so $\mathbf{x} = (-5, 10, 0, -15) - (1, 3, -2, -5) = (-6, 7, 2, -10)$.

Check. Left: $\mathbf{x} - 2\mathbf{u} = (-10, 15, 2, -22)$, times 3 is (−30, 45, 6, −66), plus $2\mathbf{v} = (2, 6, -4, -10)$ gives (−28, 51, 2, −76). Right: $5\mathbf{x} = (-30, 35, 10, -50)$ and $\mathbf{u} - 4\mathbf{v} = (-2, -16, 8, 26)$, so $5\mathbf{x} - (\mathbf{u} - 4\mathbf{v}) = (-28, 51, 2, -76)$. The two sides agree.

6. The points $\mathbf{p} = (7, -4, 1, -6)$, $\mathbf{q} = (4, 3, -1, 6)$ and $\mathbf{r} = (-2, 7, 3, 0)$ of $\mathbb{R}^4$ are the corners of a triangle.

(a) Find the midpoint $\mathbf{m}$ of the segment from $\mathbf{q}$ to $\mathbf{r}$.

(b) Find the point $\mathbf{g}$ two thirds of the way from $\mathbf{p}$ to $\mathbf{m}$.

(c) Show that for any three points $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ of any $\mathbb{R}^n$, the point two thirds of the way from $\mathbf{p}$ to the midpoint of $\mathbf{q}$ and $\mathbf{r}$ is $\frac{1}{3}(\mathbf{p} + \mathbf{q} + \mathbf{r})$, naming the properties you use. Use this to check (b).

(d) Without computing, explain why the point two thirds of the way from $\mathbf{q}$ to the midpoint of $\mathbf{p}$ and $\mathbf{r}$ is the same point $\mathbf{g}$.

ANSWER

(a) $\mathbf{m} = (1, 5, 1, 3)$ (b) $\mathbf{g} = (3, 2, 1, 0)$ (c) $\frac{1}{3}(\mathbf{p} + \mathbf{q} + \mathbf{r}) = \frac{1}{3}(9, 6, 3, 0) = (3, 2, 1, 0)$, matching (b). (d) The expression $\frac{1}{3}(\mathbf{p} + \mathbf{q} + \mathbf{r})$ does not change when the three points swap roles.

SOLUTION

(a) $\mathbf{m} = \frac{1}{2}(\mathbf{q} + \mathbf{r}) = \frac{1}{2}(2, 10, 2, 6) = (1, 5, 1, 3)$.

(b) The step from $\mathbf{p}$ to $\mathbf{m}$ is $\mathbf{m} - \mathbf{p} = (-6, 9, 0, 9)$. Two thirds of it is (−4, 6, 0, 6), so $\mathbf{g} = \mathbf{p} + \frac{2}{3}(\mathbf{m} - \mathbf{p}) = (3, 2, 1, 0)$.

(c) With $\mathbf{m} = \frac{1}{2}(\mathbf{q} + \mathbf{r})$:

$$\begin{aligned} \mathbf{p} + \tfrac{2}{3}(\mathbf{m} - \mathbf{p}) &= \mathbf{p} + \tfrac{2}{3}\mathbf{m} - \tfrac{2}{3}\mathbf{p} && \text{(properties 5 and 7)} \\ &= \tfrac{1}{3}\mathbf{p} + \tfrac{2}{3}\mathbf{m} && \text{(properties 1, 2, 8 and 6)} \\ &= \tfrac{1}{3}\mathbf{p} + \tfrac{1}{3}(\mathbf{q} + \mathbf{r}) && \text{(property 7: } \tfrac{2}{3} \cdot \tfrac{1}{2} = \tfrac{1}{3}\text{)} \\ &= \tfrac{1}{3}(\mathbf{p} + \mathbf{q} + \mathbf{r}) && \text{(properties 5 and 2)} \end{aligned}$$

Nothing here depends on n. Check: $\mathbf{p} + \mathbf{q} + \mathbf{r} = (9, 6, 3, 0)$ and $\frac{1}{3}(9, 6, 3, 0) = (3, 2, 1, 0) = \mathbf{g}$.

(d) By (c) with the roles of $\mathbf{p}$ and $\mathbf{q}$ swapped, that point is $\frac{1}{3}(\mathbf{q} + \mathbf{p} + \mathbf{r})$. Addition is commutative and associative (properties 1 and 2), so $\mathbf{q} + \mathbf{p} + \mathbf{r} = \mathbf{p} + \mathbf{q} + \mathbf{r}$, and the point is $\mathbf{g}$ again.

7. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$ and c a scalar.

(a) Prove entry by entry that $(-c)\mathbf{u} = -(c\mathbf{u})$.

(b) Prove, using only the eight properties and no entries, that if $\mathbf{u} + \mathbf{v} = \mathbf{0}$, then $\mathbf{v} = -\mathbf{u}$. Name the property behind each step.

ANSWER

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(a) Both sides have i-th entry $-cu_i$, for every i. (b) $\mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v} + (\mathbf{u} + (-\mathbf{u})) = (\mathbf{v} + \mathbf{u}) + (-\mathbf{u}) = (\mathbf{u} + \mathbf{v}) + (-\mathbf{u}) = \mathbf{0} + (-\mathbf{u}) = -\mathbf{u}$.

SOLUTION

(a) The i-th entry of $(-c)\mathbf{u}$ is $(-c)u_i$. Since $-(c\mathbf{u}) = (-1)(c\mathbf{u})$, its i-th entry is $(-1)(cu_i)$. For real numbers, $(-c)u_i = -(cu_i) = (-1)(cu_i)$. The entries agree for every $i = 1, \ldots, n$, so the vectors are equal by the definition of equality.

(b) Suppose $\mathbf{u} + \mathbf{v} = \mathbf{0}$. Then

$$\begin{aligned}
\mathbf{v} &= \mathbf{v} + \mathbf{0} && \text{(property 3)} \\
&= \mathbf{v} + (\mathbf{u} + (-\mathbf{u})) && \text{(property 4)} \\
&= (\mathbf{v} + \mathbf{u}) + (-\mathbf{u}) && \text{(property 2)} \\
&= (\mathbf{u} + \mathbf{v}) + (-\mathbf{u}) && \text{(property 1)} \\
&= \mathbf{0} + (-\mathbf{u}) && \text{(the hypothesis)} \\
&= -\mathbf{u} && \text{(properties 1 and 3)}
\end{aligned}$$

So a vector that adds to $\mathbf{u}$ to give $\mathbf{0}$ can only be $-\mathbf{u}$: each vector has exactly one negative.

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