Solving a system by row reduction

Row reduce a system and read its last rows. They tell you whether there is one solution, none, or infinitely many, and how to write them all down.

WORDS TO KNOW

Consistent: the system has at least one solution.
Inconsistent: it has none.

Leading variable: a variable whose column holds a pivot.

Parameter: a letter such as t that stands for any real number.

Pivot (leading entry): the first nonzero entry of a row in echelon form.

Free variable: a variable whose column has no pivot. It may take any value.

Reduced row echelon form: echelon form with every pivot 1 and zeros above and below each pivot.

KEY IDEAS

1. Read the last rows of the echelon form

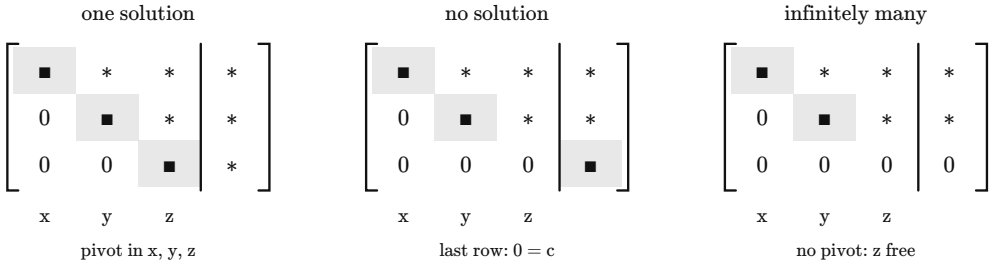
Row operations never change the solutions, so the echelon form tells the truth about the original system. Read it from the bottom up.

A row $0\ \ 0\ \ 0 \mid c$ with $c \neq 0$ says $0 = c$: **no solution**. Stop.

A row of all zeros says 0 = 0. It adds nothing; ignore it.

Otherwise the system is consistent. A pivot in every variable column: **exactly one solution**. A variable column with no pivot: that variable is free, and there are **infinitely many** solutions.

A linear system never has exactly two solutions: if it has two, it has infinitely many.



2. Write an infinite family with a parameter

Go on to reduced form. Each row then reads "leading variable plus free terms equals a number". Set each free variable equal to a parameter and solve for the leading variables.

Number of free variables = number of variable columns − number of pivots.

So a consistent system with more variables than equations always has infinitely many solutions.

3. A letter in the matrix

Carry the letter k through the row operations. Each value of k that makes a would-be pivot 0 is its own case.

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WATCH OUT

- **Never divide by an expression that might be 0.** Scaling a row by $\frac{1}{k-3}$ is allowed only when $k \neq 3$. Treat $k = 3$ as a separate case.
- **Read the rows only once you reach echelon form.** A zero row means nothing while another row could still be cleared against it.
- **The free variable need not be the last one.** Look for the column with no pivot.
- **Work with exact numbers.** A rounded decimal can leave a tiny entry where a 0 belongs.

WORKED EXAMPLES

Example 1. Solve $x + y + 2z = 3$, $2x + 3y + 3z = 7$, $3x + 4y + 5z = 8$.

1. Write the augmented matrix and clear the first column with $R_2 - 2R_1 \to R_2$ and $R_3 - 3R_1 \to R_3$.
$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right]$$
2. Clear below the second pivot with $R_3 - R_2 \to R_3$. Row 3 becomes $0 \;\; 0 \;\; 0 \mid -2$.
3. That row says $0 = -2$, which no triple makes true.

Answer: no solution. **Check:** adding the first two equations gives $3x + 4y + 5z = 10$, but the third says the same left side is 8.

Example 2. Solve $x + y - z = 2$, $2x + 3y + z = 9$, $x + 2y + 2z = 7$, and write the solution set.

1. $R_2 - 2R_1 \to R_2$ and $R_3 - R_1 \to R_3$ both give the row $0 \;\; 1 \;\; 3 \mid 5$.
2. $R_3 - R_2 \to R_3$ makes row 3 all zeros. No row says $0 = c$, so the system is consistent. The pivots are in the x and y columns, so z is free.
3. Go to reduced form with $R_1 - R_2 \to R_1$:
$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
4. The rows read $x - 4z = -3$ and $y + 3z = 5$. Let $z = t$.

Answer: $(x, y, z) = (-3 + 4t,\ 5 - 3t,\ t)$ for every real t. **Check:** $t = 1$ gives (1, 2, 1), and 1 + 2 − 1 = 2, 2 + 6 + 1 = 9, 1 + 4 + 2 = 7.

Example 3. For which k does $x + y = 1$, $2x + (k^2 - 7)y = k - 1$ have one solution, none, or infinitely many?

1. $R_2 - 2R_1 \to R_2$ gives the row $0 \;\; k^2 - 9 \mid k - 3$. No step divided by an expression in k.
2. The entry $k^2 - 9 = (k - 3)(k + 3)$ decides the case.
3. $k = 3$: the row is $0 \;\; 0 \mid 0$, so y is free. With $y = t$, $x = 1 - t$.
4. $k = -3$: the row is $0 \;\; 0 \mid -6$, which says $0 = -6$.
5. Any other k: two pivots, so one solution, with $x = 1 - y$ and
$$y = \frac{k - 3}{k^2 - 9} = \frac{1}{k + 3}$$

Answer: $k = 3$: infinitely many, $(1 - t,\ t)$. $k = -3$: no solution. Every other k: exactly one. **Check:** $k = 0$ gives $y = \frac{1}{3}$ and $x = \frac{2}{3}$, and then $2x - 7y = \frac{4}{3} - \frac{7}{3} = -1$, which is $k - 1$.

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