Linear Algebra Unit 2 worksheet: Solving a system by row reduction: one, none or infinitely many solutions (answer key)

ANSWERS AT A GLANCE

1. (a) $(x, y, z, w) = (3 - 2t,\ t,\ -2 - 4t,\ t)$ (b) No solution (c) $(x, y, z) = (s,\ 3 + 2t,\ t)$
2. $(x, y, z, w) = (1 - 2t,\ -3 - 3t,\ 1 + 2t,\ t)$ for every real t
3. Infinitely many: $(x, y, z) = (1 - 2t,\ 3 + t,\ t)$ for every real t
4. (a) $c = -3$ (b) $(x, y, z) = (17 - 5t,\ -4 + 2t,\ t)$ for every real t
5. (a) $a + b + c = 20$ and $50a + 70b + 90c = 1{,}300$; $(a, b, c) = (5 + t,\ 15 - 2t,\ t)$ (b) 8 purchases ($t = 0, 1, \ldots, 7$); the most 90-cent stamps: 12, 1 and 7 stamps.
6. $k \neq 1$ and $k \neq -2$: one solution, $\left(\frac{1}{k+2}, \frac{1}{k+2}, \frac{1}{k+2}\right)$. $k = 1$: infinitely many, $(x, y, z) = (1 - s - t,\ s,\ t)$. $k = -2$: no solution.

WORKED SOLUTIONS

1. Each matrix is the result of row reducing the augmented matrix of a system. The variables are x, y, z, w in (a) and x, y, z in (b) and (c). Write the solution set with parameters, or say that there is no solution.

(a) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 4 & -2 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 7 \end{array}\right]$ (c) $\left[\begin{array}{ccc|c} 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

ANSWER

(a) $(x, y, z, w) = (3 - 2t,\ t,\ -2 - 4t,\ t)$ (b) No solution (c) $(x, y, z) = (s,\ 3 + 2t,\ t)$

SOLUTION

(a) No row reads $0 = c$, and the w column has no pivot, so w is free. The rows read $x + 2w = 3$, $y - w = 0$ and $z + 4w = -2$. With $w = t$: $(x, y, z, w) = (3 - 2t,\ t,\ -2 - 4t,\ t)$ for every real t.

(b) The last row says 0 = 7, which no triple makes true. No solution, whatever the first two rows say.

(c) The only condition is $y - 2z = 3$. The x column is all zeros, so it holds no pivot and x is free as well as z. With $x = s$ and $z = t$: $(x, y, z) = (s,\ 3 + 2t,\ t)$ for all real s and t.

2. Solve the system in x, y, z, w whose augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & -4 & 7 \\ -3 & 7 & 2 & 11 & -22 \\ -2 & 1 & -5 & 9 & -10 \end{array}\right]$$

Reduce to reduced row echelon form and write the solution set with a parameter.

ANSWER

$(x, y, z, w) = (1 - 2t,\ -3 - 3t,\ 1 + 2t,\ t)$ for every real t

SOLUTION

Clear the first column with $R_2 + 3R_1 \to R_2$ and $R_3 + 2R_1 \to R_3$:

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & -4 & 7 \\ 0 & 1 & 2 & -1 & -1 \\ 0 & -3 & -5 & 1 & 4 \end{array}\right]$$

Then $R_3 + 3R_2 \to R_3$ gives an echelon form:

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & -4 & 7 \\ 0 & 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

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No row reads $0 = c$, so the system is consistent. The pivots are in the x, y and z columns, so w is free and there are infinitely many solutions. Clear above the pivots with $R_2 - 2R_3 \to R_2$, then $R_1 + 2R_2 \to R_1$:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 3 & -3 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

The rows read $x + 2w = 1$, $y + 3w = -3$ and $z - 2w = 1$. With $w = t$:

$$(x, y, z, w) = (1 - 2t,\ -3 - 3t,\ 1 + 2t,\ t)$$

Check with $t = 0$, the solution (1, −3, 1, 0), in the first equation: $1 - 2 \cdot (-3) = 7$.

3. Consider the system

$$\begin{aligned} x + y + z &= 4 \\ 2x + 3y + z &= 11 \\ -x + y - 3z &= 2 \\ 3x + 2y + 4z &= 9 \end{aligned}$$

Solve it by row reduction. Say how many solutions it has, and write the solution set if there is one.

ANSWER

Infinitely many: $(x, y, z) = (1 - 2t,\ 3 + t,\ t)$ for every real t

SOLUTION

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 3 & 1 & 11 \\ -1 & 1 & -3 & 2 \\ 3 & 2 & 4 & 9 \end{array}\right]$$

Clear the first column: $R_2 - 2R_1 \to R_2$ gives $0\ \ 1\ \ -1 \mid 3$, $R_3 + R_1 \to R_3$ gives $0\ \ 2\ \ -2 \mid 6$, and $R_4 - 3R_1 \to R_4$ gives $0\ \ -1\ \ 1 \mid -3$. Then $R_3 - 2R_2 \to R_3$ and $R_4 + R_2 \to R_4$ both give zero rows. Finish with $R_1 - R_2 \to R_1$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The rows read $x + 2z = 1$ and $y - z = 3$. With $z = t$: $(x, y, z) = (1 - 2t,\ 3 + t,\ t)$. **Check** in the fourth equation: $3(1 - 2t) + 2(3 + t) + 4t = 3 + 6 - 6t + 2t + 4t = 9$.

Four equations in three unknowns, yet infinitely many solutions: what counts is the number of pivots, not of equations. The two zero rows say 0 = 0 because the third equation is −5 times the first plus 2 times the second, and the fourth is 5 times the first minus the second. Two pivots for three variable columns and no row $0 = c$ leave z free.

4. The augmented matrix of a system in x, y, z is

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 2 & 7 & -4 & 6 \\ -3 & -12 & 9 & c \end{array}\right]$$

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where c is a constant.

(a) Find the value of c for which the system is consistent.

(b) For that value of c, write the solution set.

ANSWER

(a) $c = -3$ (b) $(x, y, z) = (17 - 5t,\ -4 + 2t,\ t)$ for every real t

SOLUTION

(a) Reduce with c carried along. $R_2 - 2R_1 \to R_2$ and $R_3 + 3R_1 \to R_3$ give

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 0 & 1 & -2 & -4 \\ 0 & -3 & 6 & c + 15 \end{array}\right]$$

Then $R_3 + 3R_2 \to R_3$ leaves the last row $0\ \ 0\ \ 0 \mid c + 3$. It says $0 = c + 3$, which is false unless $c = -3$. So for every other c there is no solution, and the system is consistent only for $c = -3$.

(b) With $c = -3$ the last row is all zeros. The pivots are in the x and y columns, so z is free. $R_1 - 3R_2 \to R_1$ gives the row $1\ \ 0\ \ 5 \mid 17$. The rows read $x + 5z = 17$ and $y - 2z = -4$. With $z = t$:

$$(x, y, z) = (17 - 5t,\ -4 + 2t,\ t)$$

Check with $t = 0$ in the first row: $17 + 3 \cdot (-4) = 5$.

5. Arjun buys 20 postage stamps for 13 dollars. Some cost 50 cents, some 70 cents and some 90 cents. Let a, b and c be the numbers of 50-cent, 70-cent and 90-cent stamps.

(a) Write two equations, one for the count and one for the cost in cents. Row reduce the augmented matrix and write the solution set with a parameter.

(b) A number of stamps must be a whole number and cannot be negative. How many different purchases are possible? Give the one with the most 90-cent stamps.

ANSWER

(a) $a + b + c = 20$ and $50a + 70b + 90c = 1{,}300$; $(a, b, c) = (5 + t,\ 15 - 2t,\ t)$ (b) 8 purchases ($t = 0, 1, \ldots, 7$); the most 90-cent stamps: 12, 1 and 7 stamps.

SOLUTION

(a) The count gives $a + b + c = 20$ and the cost gives $50a + 70b + 90c = 1{,}300$. Dividing the second equation by 10:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 5 & 7 & 9 & 130 \end{array}\right]$$

$R_2 - 5R_1 \to R_2$ gives $0\ \ 2\ \ 4 \mid 30$, and $\frac{1}{2}R_2 \to R_2$ gives $0\ \ 1\ \ 2 \mid 15$. Then $R_1 - R_2 \to R_1$ gives $1\ \ 0\ \ -1 \mid 5$. The rows read $a - c = 5$ and $b + 2c = 15$; c is free. With $c = t$:

$$(a, b, c) = (5 + t,\ 15 - 2t,\ t)$$

(b) $c = t \geq 0$; $a = 5 + t \geq 0$ holds for every such t; $b = 15 - 2t \geq 0$ needs $t \leq 7.5$. Whole numbers come from whole t, so $t = 0, 1, \ldots, 7$: 8 purchases. The most 90-cent stamps is $t = 7$: 12 stamps at 50 cents, 1 at 70 cents and 7 at 90 cents. **Check:** 12 + 1 + 7 = 20 and 600 + 70 + 630 = 1,300 cents.

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6. In the system

$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= 1 \\ x + y + kz &= 1 \end{aligned}$$

the letter k appears once in each equation. Start with $R_1 \leftrightarrow R_3$, so that the first pivot is not an expression in k, and reduce with k carried along. Then sort every real k by the number of solutions, and write the solution set in each case that has solutions.

ANSWER

$k \neq 1$ and $k \neq -2$: one solution, $\left(\frac{1}{k+2}, \frac{1}{k+2}, \frac{1}{k+2}\right)$. $k = 1$: infinitely many, $(x, y, z) = (1 - s - t,\ s,\ t)$. $k = -2$: no solution.

SOLUTION

After $R_1 \leftrightarrow R_3$:

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \end{array}\right]$$

$R_2 - R_1 \to R_2$ gives $0 \;\; k-1 \;\; 1-k \mid 0$, and $R_3 - kR_1 \to R_3$ gives $0 \;\; 1-k \;\; 1-k^2 \mid 1-k$. Then $R_3 + R_2 \to R_3$ gives $0 \;\; 0 \;\; 2-k-k^2 \mid 1-k$:

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & -(k-1)(k+2) & -(k-1) \end{array}\right]$$

No operation divided by an expression in k, so this holds for every k. The would-be pivots $k - 1$ and $-(k-1)(k+2)$ are 0 only at $k = 1$ and $k = -2$.

- $k = 1$: rows 2 and 3 are all zeros. Only $x + y + z = 1$ is left, y and z are free: $(x, y, z) = (1 - s - t,\ s,\ t)$, infinitely many.
- $k = -2$: row 3 is $0 \;\; 0 \;\; 0 \mid 3$, which says 0 = 3: no solution.
- Any other k: there is a pivot in every variable column, so one solution. Row 3 gives $z = \frac{-(k-1)}{-(k-1)(k+2)} = \frac{1}{k+2}$. Row 2 is $(k-1)(y-z) = 0$ with $k - 1 \neq 0$, so $y = z = \frac{1}{k+2}$. Row 1 gives $x = 1 - y - kz = 1 - \frac{1+k}{k+2} = \frac{1}{k+2}$.

Check $k = 0$: the system is $y + z = 1$, $x + z = 1$, $x + y = 1$, solved by $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

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