Cofactor expansion

How do you find the determinant of a 4 by 4 matrix, or any size? Expand along a row or column, and let the zeros do the work.

WORDS TO KNOW

Minor M_{ij}: the determinant left after you cross out row i and column j.

Checkerboard sign: the sign $(-1)^{i+j}$ for row i, column j. Every diagonal position is $+$.

Upper triangular: every entry below the main diagonal is 0. **Lower triangular:** every entry above it is 0.

Cofactor C_{ij}: the minor with the checkerboard sign attached, $C_{ij} = (-1)^{i+j} M_{ij}$.

Expansion along a line: each entry of one row (or one column) times its cofactor, all added up.

KEY IDEAS

1. The definition, one size at a time

An $n \times n$ determinant is built from $(n-1) \times (n-1)$ ones, and so on down to 1 × 1, where $\det[a] = a$. Each step lowers the size, so the definition is not circular.

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

The signs for a 4 × 4 matrix:

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

2. Any row or column gives the same number

You may expand along any row i or any column j:

$$\det A = a_{i1}C_{i1} + \cdots + a_{in}C_{in} = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj}$$

So **pick the line with the most zeros**. A zero entry adds $0 \cdot C_{ij} = 0$, so you never need its cofactor. Inside each minor, choose the best line again.

3. Triangular matrices

For an upper triangular matrix, expand down column 1 again and again (for a lower one, along row 1): only the diagonal entry survives each time.

The determinant of a triangular matrix is the product of its diagonal entries. So $\det I_n = 1$.

$$\det \begin{bmatrix} 2 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 3 & 5 & 4 & 0 \\ -6 & 1 & 9 & 3 \end{bmatrix} = 2 \cdot (-1) \cdot 4 \cdot 3 = -24$$

4. Letters and cost

When some entries are a letter, expansion gives a polynomial in it. Its zeros are the values where the determinant is 0, so the system with that matrix stops having exactly one solution. A full expansion with no zeros forms $n!$ products, which grows very fast, so expansion suits small matrices, matrices with many zeros, and matrices holding a letter.

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WATCH OUT

- **The sign belongs to the position, not the entry.** A negative entry in a $+$ position still gets $+$. Inside a minor, use the positions of the smaller matrix.
- **Expand along a whole row or a whole column, never the diagonal.** And only a square matrix has a determinant.
- **The triangular rule needs every entry on one side of the main diagonal to be 0.** A matrix whose zeros all lie on one side of the other diagonal is not triangular.

WORKED EXAMPLES

Example 1. Find $\det G$ **for the matrix below.**

$$G = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 2 & 0 & 1 & -1 \\ 4 & 1 & 2 & 0 \\ 0 & 2 & -3 & 0 \end{bmatrix}$$

1. Column 4 is (2, −1, 0, 0), with two zeros. Its signs are $-, +, -, +$ from the top, so
$$\det G = 2(-M_{14}) + (-1)(+M_{24}) = -2M_{14} - M_{24}$$
2. Cross out row 1 and column 4, and expand along row 1 of what is left:
$$M_{14} = \begin{vmatrix} 2 & 0 & 1 \\ 4 & 1 & 2 \\ 0 & 2 & -3 \end{vmatrix} = 2(-3-4) - 0 + 1(8-0) = -6$$
3. Cross out row 2 and column 4, and expand along row 1:
$$M_{24} = \begin{vmatrix} 1 & 3 & 0 \\ 4 & 1 & 2 \\ 0 & 2 & -3 \end{vmatrix} = 1(-3-4) - 3(-12-0) + 0 = 29$$
4. $\det G = -2(-6) - 29 = 12 - 29$.

Answer: $\det G = -17$.

Example 2. For which k **is the determinant of the matrix below 0?**

$$K = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & k & 0 & 0 \\ 1 & 4 & 3 & 2 \\ k & 0 & 0 & 1 \end{bmatrix}$$

1. Row 2 is $(0, k, 0, 0)$: only k, in position (2, 2), sign $+$, can be nonzero. Cross out row 2 and column 2:
$$\det K = k \begin{vmatrix} 2 & 0 & 3 \\ 1 & 3 & 2 \\ k & 0 & 1 \end{vmatrix}$$
2. In this 3 × 3, column 2 holds only the 3, in position (2, 2), sign $+$:
$$\begin{vmatrix} 2 & 0 & 3 \\ 1 & 3 & 2 \\ k & 0 & 1 \end{vmatrix} = 3 \begin{vmatrix} 2 & 3 \\ k & 1 \end{vmatrix} = 3(2 - 3k)$$
3. So $\det K = 3k(2-3k)$, which is 0 when $k = 0$ or $2 - 3k = 0$.

Answer: $k = 0$ or $k = \frac{2}{3}$. **Check:** at $k = 0$, row 2 is all zeros, and expanding along it gives 0.

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