Linear Algebra Unit 4 worksheet: Determinants by cofactor expansion (answer key)

ANSWERS AT A GLANCE

1. (a) $C_{11} = 10$ and $C_{12} = -20$ (the 0 in position $(1, 3)$ needs no cofactor), and $\det A = -10$
   (b) $C_{31} = 8$ and $C_{33} = 7$ (the 0 in position $(3, 2)$ needs no cofactor), and $-3 \cdot 8 + 2 \cdot 7 = -10$
2. $\det M = 36$, expanding first down column 3, which has only one nonzero entry.
3. (a) $\det L = 12$
   (b) $\det B = 6$
   (c) The diagonal product is -12. B is not triangular: the 3 in position $(2, 1)$ sits below the diagonal, so the triangular rule does not apply.
4. (a) $\det K = (3k - 13)(1 - k) = -3k^2 + 16k - 13$
   (b) $k = 1$ and $k = \dfrac{13}{3}$
5. $\det Q = -45$
6. (a) $n = 4$
   (b) $n = 16$
7. (a) $D_1 = 3$, $D_2 = 8$
   (b) $D_n = 3D_{n-1} - D_{n-2}$ (see the solution)
   (c) $D_6 = 377$ (from $D_3 = 21$, $D_4 = 55$, $D_5 = 144$)

WORKED SOLUTIONS

1. Let

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 4 & 5 & 4 \\ -3 & 0 & 2 \end{bmatrix}.$$

(a) Find $\det A$ by expanding along row 1. Find only the cofactors you need, and write each one down.

(b) Row 3 also has a zero. Find only the cofactors you need to expand along row 3, and check that you get the same $\det A$.

ANSWER

(a) $C_{11} = 10$ and $C_{12} = -20$ (the 0 in position (1, 3) needs no cofactor), and $\det A = -10$

(b) $C_{31} = 8$ and $C_{33} = 7$ (the 0 in position (3, 2) needs no cofactor), and $-3 \cdot 8 + 2 \cdot 7 = -10$

SOLUTION

(a) Row 1 is (3, 2, 0) with signs $+, -, +$; the 0 contributes $0 \cdot C_{13} = 0$, so C_{13} is not needed. Cross out row 1 and columns 1 and 2 in turn:

$$C_{11} = +\begin{vmatrix} 5 & 4 \\ 0 & 2 \end{vmatrix} = 10, \qquad C_{12} = -\begin{vmatrix} 4 & 4 \\ -3 & 2 \end{vmatrix} = -20$$

Then

$$\det A = 3 \cdot 10 + 2 \cdot (-20) = -10.$$

(b) Row 3 is $(-3, 0, 2)$ with signs $+, -, +$. Cross out row 3 and column 1, then row 3 and column 3:

$$C_{31} = +\begin{vmatrix} 2 & 0 \\ 5 & 4 \end{vmatrix} = 8, \qquad C_{33} = +\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 7$$

So $\det A = -3 \cdot 8 + 2 \cdot 7 = -10$, the same number, as the theorem on expansion along any line promises.

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2. Find $\det M$ for the matrix below. Say which row or column you expand along first, and why.

$$M = \begin{bmatrix} 2 & 2 & 0 & -3 \\ -1 & -4 & 2 & 2 \\ 1 & 4 & 0 & -2 \\ -3 & 0 & 0 & 1 \end{bmatrix}$$

ANSWER

$\det M = 36$, expanding first down column 3, which has only one nonzero entry.

SOLUTION

Column 3 is (0, 2, 0, 0), so only the 2 in position (2, 3) counts, with sign $(-1)^{2+3} = -$. Crossing out row 2 and column 3 leaves

$$N = \begin{bmatrix} 2 & 2 & -3 \\ 1 & 4 & -2 \\ -3 & 0 & 1 \end{bmatrix}, \qquad \det M = -2 \det N.$$

In N, row 3 is (−3, 0, 1) with signs $+, -, +$:

$$\det N = (-3)\begin{vmatrix} 2 & -3 \\ 4 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} = (-3)\cdot 8 + 1 \cdot 6 = -18$$

So $\det M = -2 \cdot (-18) = 36$.

Row 4 of M also has two zeros and gives the same number, but it needs two 3 × 3 minors instead of one.

3. (a) Find $\det L$ for

$$L = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 & 0 \\ 1 & -4 & \frac{1}{2} & 0 & 0 \\ 7 & 2 & 6 & -1 & 0 \\ -3 & 8 & 1 & 9 & 4 \end{bmatrix}$$

(b) Find $\det B$ for

$$B = \begin{bmatrix} 2 & 1 & 4 & -1 \\ 3 & 1 & 0 & 5 \\ 0 & 0 & -2 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(c) Multiply the diagonal entries of B. Why does that product not give $\det B$, and which entry is to blame?

ANSWER

(a) $\det L = 12$

(b) $\det B = 6$

(c) The diagonal product is −12. B is not triangular: the 3 in position (2, 1) sits below the diagonal, so the triangular rule does not apply.

SOLUTION

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(a) Every entry above the diagonal of L is 0, so L is lower triangular and $\det L$ is the product of the diagonal:

$$\det L = (-2)(3)\left(\tfrac{1}{2}\right)(-1)(4) = 12.$$

(b) Column 1 is (2, 3, 0, 0): the 2 has sign $+$ and the 3 in position (2, 1) has sign $-$. Both minors are upper triangular:

$$\det B = 2\begin{vmatrix} 1 & 0 & 5 \\ 0 & -2 & 6 \\ 0 & 0 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 & -1 \\ 0 & -2 & 6 \\ 0 & 0 & 3 \end{vmatrix} = 2(-6) - 3(-6) = 6.$$

(c) $2 \cdot 1 \cdot (-2) \cdot 3 = -12 \neq 6$. The product rule needs every entry below the diagonal to be 0; the 3 in position (2, 1) is not, and it brings in the second term $-3 \cdot (-6) = 18$ of the expansion above.

4. For a number k, let

$$K = \begin{bmatrix} 1 & k & 0 & 2 \\ 0 & 0 & 3 & 1 \\ 2 & 0 & 1 & k \\ 1 & 1 & 3 & 3 \end{bmatrix}.$$

(a) Find $\det K$ as a polynomial in k, and factor it.

(b) For which k does the system $K\mathbf{x} = \mathbf{b}$ fail to have exactly one solution?

ANSWER

(a) $\det K = (3k - 13)(1 - k) = -3k^2 + 16k - 13$

(b) $k = 1$ and $k = \dfrac{13}{3}$

SOLUTION

(a) Column 2 is $(k, 0, 0, 1)$: k in position (1, 2) with sign $-$, and 1 in position (4, 2) with sign $+$.

The minor of k crosses out row 1 and column 2; along its row 1,

$$M_{12} = \begin{vmatrix} 0 & 3 & 1 \\ 2 & 1 & k \\ 1 & 3 & 3 \end{vmatrix} = 0 - 3(6 - k) + 1(6 - 1) = 3k - 13.$$

The minor of the 1 crosses out row 4 and column 2; along its row 1,

$$M_{42} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 2 & 1 & k \end{vmatrix} = 1(3k - 1) - 0 + 2(0 - 6) = 3k - 13.$$

So

$$\det K = -k(3k - 13) + 1(3k - 13) = (3k - 13)(1 - k).$$

(b) The system has exactly one solution exactly when $\det K \neq 0$, so it fails for $k = 1$ and $k = \dfrac{13}{3}$.

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5. Find $\det Q$ for

$$Q = \begin{bmatrix} 3 & 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 4 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 3 & 1 \\ 0 & 1 & -1 & 0 & 2 \end{bmatrix}$$

No row or column has only one nonzero entry, so choose a line with two, and choose afresh inside each minor.

ANSWER
$\det Q = -45$

SOLUTION
Column 3 is (0, 2, 0, 0, –1): the 2 in position (2, 3) has sign $(-1)^5 = -$ and the –1 in position (5, 3) has sign $(-1)^8 = +$. So

$$\det Q = -2M_{23} + (-1)M_{53}.$$

M_{23} crosses out row 2 and column 3:

$$M_{23} = \begin{vmatrix} 3 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 3 & 1 \\ 0 & 1 & 0 & 2 \end{vmatrix}$$

Its column 4 is (0, 0, 1, 2): the 1 in position (3, 4) has sign $-$, the 2 in position (4, 4) has sign $+$.

$$\begin{vmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 3(0-1) = -3, \qquad \begin{vmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 3(6) - 1(0-2) + 2(0-4) = 12$$

(the first down its column 1). So $M_{23} = -1(-3) + 2(12) = 27$.
M_{53} crosses out row 5 and column 3:

$$M_{53} = \begin{vmatrix} 3 & 1 & 2 & 0 \\ 1 & 0 & 4 & 1 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 3 & 1 \end{vmatrix}$$

Its column 2 is (1, 0, 2, 0), both entries in $-$ positions:

$$\begin{vmatrix} 1 & 4 & 1 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{vmatrix} = 1(1-2) = -1, \qquad \begin{vmatrix} 3 & 2 & 0 \\ 1 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 3(4-3) - 2(1-2) = 5$$

(the first along its row 2). So $M_{53} = -1(-1) - 2(5) = -9$.
Then $\det Q = -2(27) + (-1)(-9) = -54 + 9 = -45$. Row 3, (0, 2, 0, 1, 0), is an equally good first line.

6. Full cofactor expansion of an $n \times n$ matrix with no zero entries forms $n!$ products, while row reduction to triangular form needs about $\frac{1}{3}n^3$ multiplications. A computer forms 10^9 products per second.

(a) Find the smallest $n \geq 2$ for which $n! > \frac{1}{3}n^3$.

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(b) What is the largest n for which full expansion on this computer finishes within one day (86,400 seconds)?

ANSWER

(a) $n = 4$

(b) $n = 16$

SOLUTION

(a) $n = 3$: $3! = 6$ and $\frac{1}{3}(27) = 9$, so $6 < 9$. $n = 4$: $4! = 24$ and $\frac{1}{3}(64) \approx 21.3$, so $24 > 21.3$. From then on $n!$ only pulls further ahead: going from n to $n+1$ multiplies $n!$ by $n+1$ but n^3 by $\left(\frac{n+1}{n}\right)^3 < 2$.

(b) In one day the computer forms $86{,}400 \times 10^9 = 8.64 \times 10^{13}$ products. $16! = 20{,}922{,}789{,}888{,}000 \approx 2.09 \times 10^{13}$ fits, and $17! = 17 \times 16! \approx 3.56 \times 10^{14}$ does not, so $n = 16$.

7. For $n \geq 1$, let D_n be the determinant of the $n \times n$ matrix with 3 in every diagonal position, 1 in every position just above or just below the diagonal, and 0 everywhere else. For example,

$$D_4 = \begin{vmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{vmatrix}.$$

(a) Find D_1 and D_2.

(b) For $n \geq 3$, expand along row 1, then expand the second minor down its column 1, to show that $D_n = 3D_{n-1} - D_{n-2}$.

(c) Find D_6.

ANSWER

(a) $D_1 = 3$, $D_2 = 8$

(b) $D_n = 3D_{n-1} - D_{n-2}$ (see the solution)

(c) $D_6 = 377$ (from $D_3 = 21$, $D_4 = 55$, $D_5 = 144$)

SOLUTION

(a) $D_1 = \det[3] = 3$. $D_2 = 3 \cdot 3 - 1 \cdot 1 = 8$.

(b) Row 1 is (3, 1, 0, ..., 0), so $D_n = 3C_{11} + 1 \cdot C_{12} = 3M_{11} - M_{12}$. Crossing out row 1 and column 1 leaves the same pattern one size smaller, so $M_{11} = D_{n-1}$. Crossing out row 1 and column 2 leaves a matrix whose column 1 is (1, 0, ..., 0): the 1 comes from position (2, 1) and the entries below it are 0. Expanding down that column, the 1 sits in a $+$ position of the smaller matrix, and crossing out its row and column leaves rows 3 to n and columns 3 to n of the original, which is the pattern of size $n - 2$. So $M_{12} = D_{n-2}$ and

$$D_n = 3D_{n-1} - D_{n-2}.$$

(c) $D_3 = 3(8) - 3 = 21$ (along row 1 directly: 3(9 – 1) – 1(3 – 0) = 21), $D_4 = 3(21) - 8 = 55$, $D_5 = 3(55) - 21 = 144$, $D_6 = 3(144) - 55 = 377$.

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