Finding eigenvectors and eigenspaces

Once you know an eigenvalue, which vectors does the matrix only stretch by that factor? Find all of them, describe them with a basis, and count the independent directions.

WORDS TO KNOW

Eigenvector for λ: a nonzero $\mathbf{v}$ with $A\mathbf{v} = \lambda\mathbf{v}$.

Eigenspace E_λ: the set $\ker(A - \lambda I)$ of all $\mathbf{x}$ with $A\mathbf{x} = \lambda\mathbf{x}$. It holds every eigenvector for λ, and also $\mathbf{0}$.

Geometric multiplicity: the dimension of E_λ, the number of independent directions λ brings.

Algebraic multiplicity: how many times λ is repeated as a root of $\det(A - \lambda I)$.

Basis of E_λ: independent vectors that span the eigenspace.

KEY IDEAS

1. An eigenspace is a kernel

$A\mathbf{v} = \lambda\mathbf{v}$ is the same as $(A - \lambda I)\mathbf{v} = \mathbf{0}$. So finding eigenvectors is solving a homogeneous system.

> For each eigenvalue λ:
>
> 1) Subtract λ from each diagonal entry of A.
>
> 2) Row reduce $A - \lambda I$ and write the solution in parametric vector form. The vectors that multiply the free variables are a basis of E_λ.
>
> 3) Check each one: $A\mathbf{v}$ must equal $\lambda\mathbf{v}$.

There is always at least one free variable. If $A - \lambda I$ reduces to I, then λ is not an eigenvalue, or the arithmetic is wrong.

2. Count the directions

> If λ is an eigenvalue, then
>
> $$1 \le \dim E_\lambda \le \text{algebraic multiplicity of } \lambda.$$

A simple root brings exactly a line. A repeated root may bring as many independent directions as it is repeated, or fewer.

3. Different eigenvalues give independent eigenvectors

> Eigenvectors for distinct eigenvalues are linearly independent. Bases of the different eigenspaces, joined together, are still independent.

So an $n \times n$ matrix has a basis of $\mathbb{R}^n$ made of its eigenvectors exactly when the geometric multiplicities add to n. Distinct real eigenvalues, n of them, always give one.

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WATCH OUT

- **0 is in every eigenspace, but it is never an eigenvector.** Do not give **0** as an answer.
- **A basis is not unique.** Any nonzero multiple of an eigenvector is another one. Your vector may differ from a friend's and both be right; one multiplication settles it.
- **Eigenvectors for the same eigenvalue can be dependent**, such as $\mathbf{v}$ and $2\mathbf{v}$. Independence is promised only across distinct eigenvalues.
- **A repeated root does not tell you the dimension of its eigenspace.** The characteristic polynomial cannot tell the cases apart; row reduce $A - \lambda I$ and count free variables.

WORKED EXAMPLES

Example 1. The matrix A has eigenvalues 7 and 1. Find a basis of each eigenspace.

$$A = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$$

1. For $\lambda = 7$, subtract 7 down the diagonal and row reduce:
$$A - 7I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$
2. So $x_1 = 2x_2$ with x_2 free, and $\mathbf{x} = x_2(2, 1)$.
3. For $\lambda = 1$, subtract 1: both rows of $A - I$ are multiples of (1, 1), so $x_1 + x_2 = 0$ and $\mathbf{x} = x_2(-1, 1)$.

Answer: $E_7 = \text{span}\{(2, 1)\}$ and $E_1 = \text{span}\{(-1, 1)\}$. **Check:** $A(2, 1) = (14, 7) = 7(2, 1)$ and $A(-1, 1) = (-1, 1)$.

Example 2. Find the eigenvalues of B with their algebraic and geometric multiplicities.

$$B = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$$

1. $\text{tr}\, B = 6$ and $\det B = 8 + 1 = 9$, so $p(\lambda) = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$. The only eigenvalue is 3, with algebraic multiplicity 2.
2. Both rows of $B - 3I$ are (1, −1), so the one equation is $x_1 = x_2$, with one free variable.

Answer: $E_3 = \text{span}\{(1, 1)\}$. Algebraic multiplicity 2, geometric multiplicity 1. **Check:** $B(1, 1) = (3, 3)$, but $B(1, 0) = (4, 1)$ is turned off its line.

Example 3. The matrix C has eigenvalues 1 (algebraic multiplicity 2) and 5. Find a basis of each eigenspace.

$$C = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix}$$

1. Rows 1 and 3 of $C - I$ are both (2, 1, 2) and row 2 is zero. The one equation $2x_1 + x_2 + 2x_3 = 0$ gives $x_2 = -2x_1 - 2x_3$, with x_1 and x_3 free:
$$\mathbf{x} = x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$
2. For $\lambda = 5$, row 2 of $C - 5I$ is $(0, -4, 0)$, so $x_2 = 0$. Then row 1, (−2, 1, 2), gives $x_1 = x_3$.

Answer: E_1 has basis (1, −2, 0), (0, −2, 1), a plane; $E_5 = \text{span}\{(1, 0, 1)\}$. **Check:** $C(0, -2, 1) = (0, -2, 1)$ and $C(1, 0, 1) = (5, 0, 5)$. The geometric multiplicities add to 2 + 1 = 3, so these three vectors are a basis of $\mathbb{R}^3$.

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