Linear Algebra Unit 5 worksheet: Finding an eigenvector (answer key)

ANSWERS AT A GLANCE

1. $E_5 = \text{span}\{(1, -3)\}$ and $E_6 = \text{span}\{(1, -2)\}$. Any nonzero multiple of either vector is accepted.
2. (a) E_4 has basis (1, 0, 1), (0, 1, −1) (any basis of the same plane is accepted); $E_7 = \text{span}\{(-2, 2, -3)\}$ (any nonzero multiple). (b) Geometric multiplicities 2 and 1. Yes: (1, 0, 1), (0, 1, −1), (−2, 2, −3) is such a basis.
3. (a) $E_2 = \text{span}\{(1, 0, 1)\}$, $E_{-1} = \text{span}\{(1, 1, 1)\}$, $E_3 = \text{span}\{(0, 1, 1)\}$ (any nonzero multiples). (b) Eigenvectors for distinct eigenvalues are independent, and three independent vectors in $\mathbb{R}^3$ form a basis.
4. (a) (2, 2, 0) for 3, (0, 1, 2) for −1 and (−1, −1, −1) for 2. The zero vector is never an eigenvector, and (1, 2, 2) is not one. (b) The eigenvalues are 3, −1 and 2, with $E_3 = \text{span}\{(1, 1, 0)\}$, $E_{-1} = \text{span}\{(0, 1, 2)\}$ and $E_2 = \text{span}\{(1, 1, 1)\}$ (any nonzero multiples).
5. (a) Vectors along the line $y = 3x$ are unchanged and vectors perpendicular to it are reversed: $E_1 = \text{span}\{(1, 3)\}$ and $E_{-1} = \text{span}\{(3, -1)\}$ (any nonzero multiples). (b) $E_1 = \text{span}\{(0, 0, 1)\}$, the axis of the turn. −1 is not an eigenvalue: no nonzero vector is reversed by a quarter turn.
6. (a) F: $E_3 = \text{span}\{(1, 0, 0), (0, 1, 0)\}$, geometric multiplicity 2; $E_1 = \text{span}\{(-1, -2, 1)\}$, multiplicity 1. G: $E_3 = \text{span}\{(1, 0, 0)\}$, geometric multiplicity 1; $E_1 = \text{span}\{(2, -2, 1)\}$, multiplicity 1. (b) F, with basis (1, 0, 0), (0, 1, 0), (−1, −2, 1). For G the geometric multiplicities add to $2 < 3$.
7. See the proof below.

WORKED SOLUTIONS

1. The matrix

$$A = \begin{bmatrix} 8 & 1 \\ -6 & 3 \end{bmatrix}$$

has eigenvalues 5 and 6. Find a basis for each eigenspace, and check each basis vector by one multiplication.

ANSWER

$E_5 = \text{span}\{(1, -3)\}$ and $E_6 = \text{span}\{(1, -2)\}$. Any nonzero multiple of either vector is accepted.

SOLUTION

For $\lambda = 5$, subtract 5 down the diagonal:

$$A - 5I = \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}.$$

Both rows are multiples of (−3, −1), so the rank is 1 and the system is the one equation $-3x_1 - x_2 = 0$ with x_1 free. The nonzero row $(a, b) = (-3, -1)$ gives the solution $(-b, a) = (1, -3)$, so $E_5 = \text{span}\{(1, -3)\}$.

For $\lambda = 6$:

$$A - 6I = \begin{bmatrix} 2 & 1 \\ -6 & -3 \end{bmatrix}.$$

Both rows are multiples of (−2, −1), so the system is $-2x_1 - x_2 = 0$, and $(-b, a)$ gives $E_6 = \text{span}\{(1, -2)\}$.

Check: $A(1, -3) = (5, -15) = 5(1, -3)$ and $A(1, -2) = (6, -12) = 6(1, -2)$.

2. The matrix

$$A = \begin{bmatrix} 10 & -6 & -6 \\ -6 & 10 & 6 \\ 9 & -9 & -5 \end{bmatrix}$$

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has eigenvalues 4, with algebraic multiplicity 2, and 7.

(a) Find a basis for each eigenspace.

(b) Give the geometric multiplicity of each eigenvalue, and decide whether $\mathbb{R}^3$ has a basis made of eigenvectors of A.

ANSWER

(a) E_4 has basis (1, 0, 1), (0, 1, −1) (any basis of the same plane is accepted); $E_7 = \text{span}\{(-2, 2, -3)\}$ (any nonzero multiple). (b) Geometric multiplicities 2 and 1. Yes: (1, 0, 1), (0, 1, −1), (−2, 2, −3) is such a basis.

SOLUTION

(a) For $\lambda = 4$:

$$A - 4I = \begin{bmatrix} 6 & -6 & -6 \\ -6 & 6 & 6 \\ 9 & -9 & -9 \end{bmatrix}.$$

Every row is a nonzero multiple of (−1, 1, 1), so the matrix reduces to one nonzero row: rank 1, two free variables. Solving that one equation for x_3 gives $x_3 = x_1 - x_2$ with x_1 and x_2 free:

$$\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

So (1, 0, 1), (0, 1, −1) is a basis of E_4, a plane. (The RREF, with its pivot in column 1, gives a different basis of the same plane, which is just as good.)

For $\lambda = 7$, a simple root, E_7 is a line, and row reducing $A - 7I$ leaves one free variable. A quicker route: the matrix above is $3\,\mathbf{u}\begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$ with $\mathbf{u} = (-2, 2, -3)$, and $\begin{bmatrix} -1 & 1 & 1 \end{bmatrix}\mathbf{u} = 1$, so $A\mathbf{u} = 4\mathbf{u} + 3\mathbf{u} = 7\mathbf{u}$. Check by multiplying: $A\mathbf{u} = (-14, 14, -21) = 7(-2, 2, -3)$. So $E_7 = \text{span}\{(-2, 2, -3)\}$.

(b) The geometric multiplicities are 2 and 1. They add to 3, so by the joined-bases result the three vectors (1, 0, 1), (0, 1, −1), (−2, 2, −3) are independent, and three independent vectors in $\mathbb{R}^3$ are a basis of eigenvectors.

3. The matrix

$$A = \begin{bmatrix} -1 & -3 & 3 \\ -4 & -1 & 4 \\ -4 & -3 & 6 \end{bmatrix}$$

has eigenvalues 2, −1 and 3.

(a) Find a basis for each eigenspace.

(b) Explain why the three vectors found in (a) together form a basis of $\mathbb{R}^3$.

ANSWER

(a) $E_2 = \text{span}\{(1, 0, 1)\}$, $E_{-1} = \text{span}\{(1, 1, 1)\}$, $E_3 = \text{span}\{(0, 1, 1)\}$ (any nonzero multiples). (b) Eigenvectors for distinct eigenvalues are independent, and three independent vectors in $\mathbb{R}^3$ form a basis.

SOLUTION

(a) For $\lambda = 2$: divide row 1 of $A - 2I$ by −3, add 4 times it to rows 2 and 3, then clear upward:

$$A - 2I = \begin{bmatrix} -3 & -3 & 3 \\ -4 & -3 & 4 \\ -4 & -3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

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So $x_1 = x_3$, $x_2 = 0$, and $E_2 = \text{span}\{(1, 0, 1)\}$.

For $\lambda = -1$: swap rows 1 and 2 of $A + I$ and scale, then clear column 1 and use row 2 to clear column 2:

$$A + I = \begin{bmatrix} 0 & -3 & 3 \\ -4 & 0 & 4 \\ -4 & -3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $x_1 = x_3$, $x_2 = x_3$, and $E_{-1} = \text{span}\{(1, 1, 1)\}$.

For $\lambda = 3$: row 3 of $A - 3I$ equals row 1, and row 2 minus row 1 is (0, −1, 1):

$$A - 3I = \begin{bmatrix} -4 & -3 & 3 \\ -4 & -4 & 4 \\ -4 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $x_1 = 0$, $x_2 = x_3$, and $E_3 = \text{span}\{(0, 1, 1)\}$.

Check: $A(1, 0, 1) = (2, 0, 2)$, $A(1, 1, 1) = (-1, -1, -1)$, $A(0, 1, 1) = (0, 3, 3)$.

(b) The three vectors are eigenvectors for the distinct eigenvalues 2, −1, 3, so they are linearly independent. Three independent vectors in $\mathbb{R}^3$ span it, so they are a basis.

4. Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \\ 6 & -6 & 2 \end{bmatrix}.$$

(a) Five vectors are offered: (0, 0, 0), (2, 2, 0), (1, 2, 2), (0, 1, 2) and (−1, −1, −1). Multiply to test them: which are eigenvectors of A, and for which eigenvalues?

(b) Without the characteristic polynomial, list every eigenvalue of A and give a basis for each eigenspace. Explain why nothing is missing.

ANSWER

(a) (2, 2, 0) for 3, (0, 1, 2) for −1 and (−1, −1, −1) for 2. The zero vector is never an eigenvector, and (1, 2, 2) is not one. (b) The eigenvalues are 3, −1 and 2, with $E_3 = \text{span}\{(1, 1, 0)\}$, $E_{-1} = \text{span}\{(0, 1, 2)\}$ and $E_2 = \text{span}\{(1, 1, 1)\}$ (any nonzero multiples).

SOLUTION

(a) $A(0, 0, 0) = (0, 0, 0)$, but the zero vector is never called an eigenvector. $A(2, 2, 0) = (2 + 4,\ 4 + 2,\ 12 - 12) = (6, 6, 0) = 3(2, 2, 0)$. $A(1, 2, 2) = (1 + 4 - 2,\ 2 + 2 - 2,\ 6 - 12 + 4) = (3, 2, -2)$, which is not a multiple of (1, 2, 2) (a multiple of (1, 2, 2) has equal second and third entries). $A(0, 1, 2) = (2 - 2,\ 1 - 2,\ -6 + 4) = (0, -1, -2) = -1(0, 1, 2)$. $A(-1, -1, -1) = (-1 - 2 + 1,\ -2 - 1 + 1,\ -6 + 6 - 2) = (-2, -2, -2) = 2(-1, -1, -1)$.

(b) Part (a) found three distinct eigenvalues 3, −1 and 2. The characteristic polynomial of a 3 × 3 matrix has degree 3, so it has at most three roots counted with multiplicity: these are all the eigenvalues, each a simple root. A simple root has $1 \le \dim E_\lambda \le 1$, so each eigenspace is the line through the eigenvector found: $E_3 = \text{span}\{(1, 1, 0)\}$, $E_{-1} = \text{span}\{(0, 1, 2)\}$, $E_2 = \text{span}\{(1, 1, 1)\}$.

5. Use the geometry first, then the matrix.

(a) The reflection of $\mathbb{R}^2$ across the line $y = 3x$ has matrix

$$R = \frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}.$$

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Which vectors does the reflection leave unchanged, and which does it reverse? Name the eigenvalues and eigenspaces this predicts, then confirm them by row reducing $R - I$ and $R + I$.

(b) The quarter turn of $\mathbb{R}^3$ about the x_3-axis (counterclockwise, seen from above) has matrix

$$Q = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find E_1 by row reduction and describe it geometrically. Then decide whether -1 is an eigenvalue of Q, and say what that means for the turn.

ANSWER

(a) Vectors along the line $y = 3x$ are unchanged and vectors perpendicular to it are reversed: $E_1 = \text{span}\{(1, 3)\}$ and $E_{-1} = \text{span}\{(3, -1)\}$ (any nonzero multiples). (b) $E_1 = \text{span}\{(0, 0, 1)\}$, the axis of the turn. -1 is not an eigenvalue: no nonzero vector is reversed by a quarter turn.

SOLUTION

(a) A reflection fixes every vector on its mirror line, so the direction $(1, 3)$ of $y = 3x$ should be an eigenvector for 1. It reverses every vector perpendicular to the line, so $(3, -1)$ should be an eigenvector for -1.

$R - I = \frac{1}{5}\begin{bmatrix} -9 & 3 \\ 3 & -1 \end{bmatrix}$: the first row is -3 times the second, so the system is $3x_1 - x_2 = 0$, giving $x_2 = 3x_1$ and $E_1 = \text{span}\{(1, 3)\}$.

$R + I = \frac{1}{5}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$: the second row is 3 times the first, so the system is $x_1 + 3x_2 = 0$, giving $E_{-1} = \text{span}\{(-3, 1)\} = \text{span}\{(3, -1)\}$.

Check: $R(1, 3) = \frac{1}{5}(5, 15) = (1, 3)$ and $R(3, -1) = \frac{1}{5}(-15, 5) = (-3, 1)$.

(b) $Q - I = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (scale row 1 by -1, subtract it from row 2, scale row 2 by $-\frac{1}{2}$, clear upward). So $x_1 = x_2 = 0$ with x_3 free: $E_1 = \text{span}\{(0, 0, 1)\}$, the x_3-axis, the only line the turn leaves in place.

$Q + I = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ reduces to I (row 2 minus row 1 is $(0, 2, 0)$), so $\ker(Q + I) = \{\mathbf{0}\}$ and -1 is not an eigenvalue: the quarter turn sends no nonzero vector to its opposite.

6. Both matrices below are upper triangular with characteristic polynomial $(3 - \lambda)^2(1 - \lambda)$:

$$F = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) For each matrix, find a basis for E_3 and for E_1, and give the geometric multiplicity of each eigenvalue.

(b) Which of the two matrices has a basis of $\mathbb{R}^3$ made of its eigenvectors? Give one such basis, and explain why the other matrix has none.

ANSWER

(a) F: $E_3 = \text{span}\{(1, 0, 0), (0, 1, 0)\}$, geometric multiplicity 2; $E_1 = \text{span}\{(-1, -2, 1)\}$, multiplicity 1. G: $E_3 = \text{span}\{(1, 0, 0)\}$, geometric multiplicity 1; $E_1 = \text{span}\{(2, -2, 1)\}$, multiplicity 1. (b) F, with basis

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$(1, 0, 0)$, $(0, 1, 0)$, $(-1, -2, 1)$. For G the geometric multiplicities add to $2 < 3$.

SOLUTION

(a) For F: $F - 3I = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & -2 \end{bmatrix}$ has rank 1, its one equation is $x_3 = 0$, and x_1, x_2 are free: E_3 has basis $(1, 0, 0)$, $(0, 1, 0)$, dimension 2. Next $F - I = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ gives $x_1 = -x_3$ and $x_2 = -2x_3$, so $E_1 = \text{span}\{(-1, -2, 1)\}$. Check: $F(-1, -2, 1) = (-3 + 2,\ -6 + 4,\ 1) = (-1, -2, 1)$.

For G: $G - 3I = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & -2 \end{bmatrix}$ has rank 2, forcing $x_2 = x_3 = 0$ with only x_1 free: $E_3 = \text{span}\{(1, 0, 0)\}$, dimension 1, below the algebraic multiplicity 2. Next $G - I = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ gives $x_2 = -2x_3$ and $x_1 = -x_2 = 2x_3$, so $E_1 = \text{span}\{(2, -2, 1)\}$. Check: $G(2, -2, 1) = (6 - 4,\ -6 + 4,\ 1) = (2, -2, 1)$.

(b) For F the geometric multiplicities are 2 + 1 = 3, so by the joined-bases result $(1, 0, 0)$, $(0, 1, 0)$, $(-1, -2, 1)$ are independent, a basis of $\mathbb{R}^3$. For G they are 1 + 1 = 2, so no independent set of eigenvectors of G has three vectors. The polynomial is the same; only the eigenspaces tell the matrices apart.

7. Let $\mathbf{v}$ be an eigenvector of the $n \times n$ matrix A for the eigenvalue λ.

(a) Suppose $\mathbf{v}$ is also an eigenvector of a second $n \times n$ matrix B, for the eigenvalue μ. Prove that $\mathbf{v}$ is an eigenvector of AB and of $A + B$, and name the eigenvalue in each case.

(b) Suppose A is invertible. Prove that $\lambda \neq 0$ and that $\mathbf{v}$ is an eigenvector of A^{-1} for $\dfrac{1}{\lambda}$.

ANSWER

(a) $AB\mathbf{v} = \lambda\mu\mathbf{v}$ and $(A + B)\mathbf{v} = (\lambda + \mu)\mathbf{v}$, with $\mathbf{v} \neq \mathbf{0}$: eigenvalues $\lambda\mu$ and $\lambda + \mu$. (b) $\lambda = 0$ would put $\mathbf{v} \neq \mathbf{0}$ in $\ker A$; then $A^{-1}\mathbf{v} = \dfrac{1}{\lambda}\mathbf{v}$.

SOLUTION

(a) We have $A\mathbf{v} = \lambda\mathbf{v}$, $B\mathbf{v} = \mu\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$. Then

$$AB\mathbf{v} = A(\mu\mathbf{v}) = \mu A\mathbf{v} = \mu\lambda\mathbf{v},$$

using that a matrix commutes with a scalar multiple, and

$$(A + B)\mathbf{v} = A\mathbf{v} + B\mathbf{v} = (\lambda + \mu)\mathbf{v}.$$

Since $\mathbf{v} \neq \mathbf{0}$, it is an eigenvector of AB for $\lambda\mu$ and of $A + B$ for $\lambda + \mu$.

(b) If $\lambda = 0$, then $A\mathbf{v} = \mathbf{0}$ with $\mathbf{v} \neq \mathbf{0}$, so $\ker A \neq \{\mathbf{0}\}$ and A would not be invertible. So $\lambda \neq 0$. Multiply $A\mathbf{v} = \lambda\mathbf{v}$ on the left by A^{-1}: $\mathbf{v} = \lambda A^{-1}\mathbf{v}$. Dividing by $\lambda \neq 0$ gives $A^{-1}\mathbf{v} = \dfrac{1}{\lambda}\mathbf{v}$, and $\mathbf{v} \neq \mathbf{0}$, so $\mathbf{v}$ is an eigenvector of A^{-1} for $\dfrac{1}{\lambda}$.

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