GCF and LCM from prime factorizations

Listing factors and multiples gets slow for big numbers. How can the primes of two numbers give you both the GCF and the LCM at once?

WORDS TO KNOW

Prime factorization: a number written as a product of primes, such as $44 = 2^2 \cdot 11$.

Power of a prime: the prime with an exponent that counts how many times it appears. In 44, the power of 2 is 2^2. A prime written alone, like 11, is 11^1.

GCF: the greatest common factor, the largest number that divides both.

LCM: the least common multiple, the smallest positive number both divide.

KEY IDEAS

1. Line up the powers of each prime

Factorize both numbers. Make a table with one column for each prime that appears in either number. Under it, write its power in each number. Write "none" if a number does not have that prime.

2. GCF takes the smaller power, LCM takes the greater

GCF: for each prime the numbers share, take the smaller power. Leave out a prime that is in only one number.

LCM: for every prime in either number, take the greater power. A prime in only one number comes in at its own power.

Why? A factor of both numbers can use a prime only as often as both of them do. A multiple of both must hold everything each number has.

3. For two numbers, GCF times LCM is the product

$$\mathrm{GCF}(a, b) \cdot \mathrm{LCM}(a, b) = a \cdot b$$

Each power goes either to the GCF or to the LCM. So if you know the GCF, divide one number by it, then multiply by the other number to get the LCM. This rule is for two numbers only. For three numbers it usually fails.

4. Which one does the question want?

Splitting into equal parts, as large as possible: the GCF. It is no bigger than the smaller number.

Lining up again, or the smallest number both go into: the LCM. It is no smaller than the larger number.

WATCH OUT

- **Do not forget a prime that is in only one number.** It must go into the LCM, or the LCM comes out too small.
- **Do not put such a prime in the GCF.** A prime that one number does not have cannot be in a common factor.
- **Smaller for GCF, greater for LCM.** Mix them up and your "GCF" may not divide both numbers. Check that it does.

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WORKED EXAMPLES

Example 1. Find the GCF and the LCM of 207 and 276.

prime	2	3	23
in 207	none	3^2	23^1
in 276	2^2	3^1	23^1

1. Factorize: $207 = 9 \cdot 23 = 3^2 \cdot 23$ and $276 = 4 \cdot 69 = 2^2 \cdot 3 \cdot 23$. Line up the powers, as in the table.
2. GCF: the shared primes are 3 and 23. Smaller powers: 3^1 and 23^1. So GCF = $3 \cdot 23 = 69$.
3. LCM: greater powers of 2, 3 and 23: $2^2 \cdot 3^2 \cdot 23 = 36 \cdot 23 = 828$.

Answer: GCF = 69 and LCM = 828. **Check:** $69 \cdot 828 = 57{,}132$ and $207 \cdot 276 = 57{,}132$.

Example 2. Two gears have 145 and 174 teeth. A marked tooth on one sits in a marked gap on the other. How many teeth pass before the marks meet again, and how many turns does each gear make?

1. The marks meet when both gears have made whole turns. So the number of teeth is a multiple of 145 and of 174: the LCM.
2. Factorize: $145 = 5 \cdot 29$ and $174 = 2 \cdot 87 = 2 \cdot 3 \cdot 29$.
3. Greater powers: 2^1, 3^1, 5^1 and 29^1. So LCM = $2 \cdot 3 \cdot 5 \cdot 29 = 30 \cdot 29 = 870$.
4. Turns: $870 \div 145 = 6$ and $870 \div 174 = 5$.

Answer: 870 teeth pass. The 145 tooth gear turns 6 times and the 174 tooth gear turns 5 times.

Example 3. GCF(66, 154) = 22. Use the product rule to find LCM(66, 154).

1. The LCM is the product divided by the GCF. Divide first to keep the numbers small: $66 \div 22 = 3$.
2. Then LCM = $3 \cdot 154 = 462$.

Answer: LCM(66, 154) = 462. **Check:** $462 \div 66 = 7$ and $462 \div 154 = 3$, both exact.

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