The sieve of Eratosthenes

You want every prime up to 120. Is there a way to find them all at once, without testing each number on its own?

WORDS TO KNOW

Prime: a whole number greater than 1 with exactly two factors, 1 and itself. 7 is prime.

Sieve: a way to find every prime up to a chosen number by crossing out the numbers that are not prime.

Smallest prime factor: the smallest prime that divides a number. For 15 it is 3.

Composite: a whole number greater than 1 with more than two factors. 8 is composite.

Round: one pass of the sieve. In the round for a prime, you cross out its multiples.

KEY IDEAS

1. Cross out the multiples of each prime, in order

Write the numbers from 1 up to the end of your board. Cross out 1: it has only one factor, so it is not prime (and not composite either).

> Circle the first number not yet crossed out. It is prime.
>
> Cross out all its multiples after it: they are composite.
>
> Repeat with the next number not crossed out.

You never need a round for 4 or 9. Every multiple of 4 is a multiple of 2, and every multiple of 9 is a multiple of 3, so those numbers are already out.

2. The round for p starts at $p \times p$

In the round for 7, the numbers 14, 21, 28, 35 and 42 are already out. Each is a multiple of a smaller prime too. The first new number is $7 \times 7 = 49$.

> A number goes out only once, in the round of its smallest prime factor.

3. Stop when $p \times p$ passes the end

A composite number on the board is the product of two whole numbers bigger than 1, and the smaller one times itself is no bigger than the number. So some small prime catches it.

> Run a round for each prime p with $p \times p$ no bigger than the last number on the board. Stop at the first prime where $p \times p$ is too big. Everything still not crossed out is prime.
>
> To test one number, try only the primes p with $p \times p$ no bigger than that number.

WATCH OUT

- **1 is not prime.** It has one factor, not two. If 1 had a round, it would cross out every number.
- **Do not stop too soon.** "Odd, digit sum not a multiple of 3, does not end in 0 or 5" only rules out 2, 3 and 5. You must keep testing until $p \times p$ is too big.
- **The sieve finds primes. It does not split a number into primes.** The round that crosses a number out tells you only its smallest prime factor.

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WORKED EXAMPLES

Example 1. Which rounds does a board from 1 to 130 need? What does the last round cross out?

1. $7 \times 7 = 49$, which is 130 or less. $11 \times 11 = 121$, also 130 or less.
2. The next prime is 13, and $13 \times 13 = 169$ is more than 130. Stop there.
3. Round 11 starts at 121. The next multiple, $11 \times 13 = 143$, is off the board.

Answer: rounds 2, 3, 5, 7 and 11. Round 11 crosses out only 121.

Example 2. On the board from 1 to 120, which round crosses out (a) 95, (b) 105, (c) 104?

1. Try 2, then 3, then 5, then 7. The first one that divides is the round.
2. (a) 95 is odd. Its digit sum 14 is not a multiple of 3. It ends in 5: $95 = 5 \times 19$.
3. (b) 105 is odd. Its digit sum is 6, a multiple of 3: $105 = 3 \times 35$. It is also a multiple of 5, but round 3 comes first.
4. (c) 104 is even.

Answer: (a) round 5 (b) round 3 (c) round 2.

Example 3. Is 109 prime? Is 299 prime?

1. For 109: $11 \times 11 = 121$ is more than 109, so test only 2, 3, 5, 7.
2. 109 is odd, its digit sum 10 is not a multiple of 3, and it does not end in 0 or 5. $7 \times 15 = 105$ leaves 4. So 109 is prime.
3. For 299: $17 \times 17 = 289$ is 299 or less, and $19 \times 19 = 361$ is more. Test 2, 3, 5, 7, 11, 13, 17.
4. 2, 3 and 5 fail. $7 \times 42 = 294$ leaves 5, and $11 \times 27 = 297$ leaves 2. But $13 \times 23 = 299$.

Answer: 109 is prime. 299 is composite: $299 = 13 \times 23$.

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