Math 6 Unit 1 quiz 3: Finding them from prime factorizations, Perfect, abundant, deficient and amicable numbers (answer key)

8 questions, 21 points

ANSWERS AT A GLANCE

1. GCF(84, 126) = 42
2. The number is 4 × 31 = 124, and it is deficient.
3. For a prime p, $s(p) = 1$, so its only possible partner is 1. But $s(1) = 0$, not p, so the sum from 1 never leads back.
4. GCF = 50, LCM = 2,100
5. (a) 63 (b) 1, 2, 3, 6, 9, 18, 27; $s(54) = 66$ (c) 1 and 2 (the new list is 3, 6, 9, 18, 27), so Leah's total is 3 short.
6. (a) $s(9) = 4$, $s(27) = 13$, $s(81) = 40$ (b) $2 \times s(n)$ is always one less than n: 8, 26, 80 against 9, 27, 81. (c) $s(243) = 121$
7. Posts 10 m apart; 46 posts in all.
8. (a) No: the LCM has only 2^3, but every number divides the LCM, so neither number can hold 2^4. (b) Four pairs: 60 and 2,520; 120 and 1,260; 180 and 840; 360 and 420.

WORKED SOLUTIONS

1. The LCM of 84 and 126 is 252. Use the product rule to find their GCF. Show your division. (2 points)

ANSWER

GCF(84, 126) = 42

SOLUTION

The product rule says GCF · LCM = 84 · 126, so the GCF is the product divided by the LCM. Divide first: 252 ÷ 84 = 3, so

$$\text{GCF} = \frac{84 \cdot 126}{252} = \frac{126}{3} = 42$$

Check: $84 = 2^2 \cdot 3 \cdot 7$ and $126 = 2 \cdot 3^2 \cdot 7$, so GCF = $2 \cdot 3 \cdot 7$ = 42.

SCORING

1 point: sets up GCF as the product of the numbers divided by the LCM. 1 point: GCF = 42.

2. A number is 4 times a prime bigger than 4, and its proper factors add to 100. Find the number, showing how you found it. Is it perfect, abundant or deficient? (2 points)

ANSWER

The number is 4 × 31 = 124, and it is deficient.

SOLUTION

Call the prime p. By unique factorization the factors of $4 \times p$ are 1, 2, 4 and p, $2 \times p$, $4 \times p$. Leaving out the number itself, the proper factors are $1, 2, 4, p$ and $2 \times p$, which add to 7 plus three times the prime.

So three times the prime is 100 − 7 = 93, and the prime is 93 ÷ 3 = 31. The number is 4 × 31 = 124.

Check: 1 + 2 + 4 + 31 + 62 = 100. Since 100 < 124, the number is deficient.

SCORING

1 point: the proper factors are 1, 2, 4, the prime and twice the prime, so 7 plus three times the prime is 100. 1 point: the number 124, deficient.

3. Explain why no prime number can belong to an amicable pair. (2 points)

ANSWER

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For a prime p, $s(p) = 1$, so its only possible partner is 1. But $s(1) = 0$, not p, so the sum from 1 never leads back.

SOLUTION

A prime p has only the factors 1 and p, so its only proper factor is 1 and $s(p) = 1$. If p were in an amicable pair with b, then $b = s(p) = 1$. The pair also needs $s(1) = p$. But 1 has no proper factors, so $s(1) = 0$, and $0 \neq p$ because p is at least 2. So the second condition always fails, and no prime is in an amicable pair.

SCORING

1 point: $s(p) = 1$, so the partner would have to be 1. 1 point: $s(1) = 0 \neq p$, so the pair fails.

4. Find the GCF and the LCM of 150 and 700. Show the prime factorizations you use. (3 points)

ANSWER

GCF = 50, LCM = 2,100

SOLUTION

$$150 = 2^1 \cdot 3 \cdot 5^2 \qquad 700 = 2^2 \cdot 5^2 \cdot 7$$

The shared primes are 2 and 5. The 3 is only in 150 and the 7 only in 700, so neither goes into the GCF. Smaller powers of the shared primes:

$$\text{GCF} = 2^1 \cdot 5^2 = 50$$

Greater power of every prime in either number:

$$\text{LCM} = 2^2 \cdot 3 \cdot 5^2 \cdot 7 = 2{,}100$$

SCORING

1 point: both prime factorizations correct. 1 point: GCF correct (smaller powers of the shared primes 2 and 5 only). 1 point: LCM correct (greater powers, with the 3 and the 7 included).

5. $s(18) = 21$. Leah says that since 54 = 3 × 18, the sum $s(54)$ must be 3 × 21. (3 points)

(a) Find 3 × 21.

(b) List the proper factors of 54 and find $s(54)$.

(c) Multiply each proper factor of 18 by 3. Which proper factors of 54 are not on that new list?

ANSWER

(a) 63 (b) 1, 2, 3, 6, 9, 18, 27; $s(54) = 66$ (c) 1 and 2 (the new list is 3, 6, 9, 18, 27), so Leah's total is 3 short.

SOLUTION

(a) 3 × 21 = 63.

(b) 54 = 1 × 54 = 2 × 27 = 3 × 18 = 6 × 9, so the proper factors are 1, 2, 3, 6, 9, 18, 27 and

$$s(54) = 1 + 2 + 3 + 6 + 9 + 18 + 27 = 66$$

(c) The proper factors of 18 are 1, 2, 3, 6, 9. Times 3 they give 3, 6, 9, 18, 27, which add to 63. Every number on that list is a proper factor of 54, but 54 also has the proper factors 1 and 2, which are not 3

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times anything. That is why $s(54) = 63 + 1 + 2 = 66$ and not 63: Leah is wrong, and $s(54)$ is more than $3 \times s(18)$.

SCORING
1 point: (a) 63. 1 point: (b) the complete list and $s(54) = 66$. 1 point: (c) names 1 and 2 as the missing factors.

6. (3 points)

(a) Find $s(9)$, $s(27)$ and $s(81)$.

(b) For each of the three, compare $2 \times s(n)$ with n. What pattern do you see?

(c) Use the pattern to predict $s(243)$, where $243 = 3^5$. Then check it by adding the proper factors.

ANSWER
(a) $s(9) = 4$, $s(27) = 13$, $s(81) = 40$ (b) $2 \times s(n)$ is always one less than n: 8, 26, 80 against 9, 27, 81. (c) $s(243) = 121$

SOLUTION
(a) A factor of a power of 3 can contain no prime but 3, so the proper factors of 3^k are the smaller powers of 3.

$$s(9) = 1 + 3 = 4 \qquad s(27) = 1 + 3 + 9 = 13 \qquad s(81) = 1 + 3 + 9 + 27 = 40$$

(b) 2 × 4 = 8 = 9 − 1, 2 × 13 = 26 = 27 − 1, 2 × 40 = 80 = 81 − 1. Twice the sum is one less than the number.

(c) The pattern predicts $2 \times s(243) = 242$, so $s(243) = 121$. Check: the proper factors are 1, 3, 9, 27, 81, and 1 + 3 + 9 + 27 + 81 = 121. (So every power of 3 is deficient: its proper factors add to less than half of it.)

SCORING
1 point: all three sums in (a). 1 point: the pattern $2 \times s(n) = n - 1$. 1 point: $s(243) = 121$ with the check by adding.

7. Diego is fencing a rectangular paddock 90 m wide and 140 m long. There is a post at each corner, and along every side the posts are the same whole number of meters apart, with the same spacing on all four sides. Diego wants to use as few posts as possible. How far apart should the posts be, and how many posts are needed in all? (3 points)

ANSWER
Posts 10 m apart; 46 posts in all.

SOLUTION
A short side runs from corner post to corner post, so the spacing must go evenly into 90, and in the same way into 140: it is a common factor. Fewer posts means a wider spacing, so the spacing is the GCF. Factorize:

$$90 = 2 \cdot 3^2 \cdot 5 \qquad 140 = 2^2 \cdot 5 \cdot 7$$

Take the smaller power of each prime the two numbers share: GCF = 10, so the posts are 10 m apart. A short side then has 90 ÷ 10 = 9 gaps and a long side 140 ÷ 10 = 14 gaps. Going once around the paddock, every gap ends at a new post, so the posts number the same as the gaps: 2 · 9 + 2 · 14 = 46 posts. Size check: 10 is no larger than 90.

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SCORING
1 point: recognizes a GCF and shows both factorizations. 1 point: the spacing. 1 point: the number of posts.

8. Two numbers have GCF $2^2 \cdot 3 \cdot 5$ and LCM $2^3 \cdot 3^2 \cdot 5 \cdot 7$. (3 points)

(a) Can one of the numbers be $2^4 \cdot 3 \cdot 5$? Explain.

(b) For each prime, one of the two numbers holds the power in the GCF and the other holds the power in the LCM. Use this to find every pair of numbers with this GCF and this LCM. Give each number as an ordinary number.

ANSWER
(a) No: the LCM has only 2^3, but every number divides the LCM, so neither number can hold 2^4. (b) Four pairs: 60 and 2,520; 120 and 1,260; 180 and 840; 360 and 420.

SOLUTION
(a) No. The LCM takes the greater power of each prime from the two numbers, so the power of 2 in either number is at most the power in the LCM, which is 2^3. The number $2^4 \cdot 3 \cdot 5$ has four 2s, so it cannot be one of them. (Equivalently, the LCM is a multiple of each number, and $2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2{,}520$ is not a multiple of $2^4 \cdot 3 \cdot 5 = 240$.)

(b) The 5 has the same power, 5^1, in the GCF and the LCM, so both numbers hold 5^1. For the other primes the powers differ: one number holds 2^3 and the other 2^2; one holds 3^2 and the other 3^1; one holds 7^1 and the other no 7. Call the number with 2^3 the first. The 3^2 can go to the first or the second, and so can the 7, which gives four pairs: - first gets 3^2 and 7: $2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2{,}520$ and $2^2 \cdot 3 \cdot 5 = 60$ - first gets 3^2 only: $2^3 \cdot 3^2 \cdot 5 = 360$ and $2^2 \cdot 3 \cdot 5 \cdot 7 = 420$ - first gets 7 only: $2^3 \cdot 3 \cdot 5 \cdot 7 = 840$ and $2^2 \cdot 3^2 \cdot 5 = 180$ - first gets neither: $2^3 \cdot 3 \cdot 5 = 120$ and $2^2 \cdot 3^2 \cdot 5 \cdot 7 = 1{,}260$

Check with the product rule: every pair multiplies to $60 \cdot 2{,}520 = 151{,}200$, for example $360 \cdot 420 = 151{,}200$.

SCORING
1 point: (a) no, with a reason about the power of 2 (or that the LCM is not a multiple of 240). 1 point: (b) both numbers hold 5^1, and for 2, 3 and 7 one number takes the GCF power and the other the LCM power. 1 point: (b) all four pairs and no others.

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