Math 6 Unit 1 test: Factors, multiples, GCF and LCM (answer key)

13 questions, 45 points

ANSWERS AT A GLANCE

1. (a) 3 blocks: 474,747 (b) 9 blocks
2. (a) No: 195 = 3 × 65, so it goes out in round 3. (b) An even lantern goes out in round 2 and a multiple of 3 in round 3, both before round 5, and a lantern goes out only once. (c) 155, 175 and 185
3. (a) 121 = 11 × 11 (b) 529 = 23 × 23 (c) A composite number is $a \times b$ with a and b greater than 1; each has a prime factor of at least 11, so each is at least 11 and the product is at least 121.
4. (a) 4 + 0 + 9 + 5 = 18, a multiple of 9, so 9 is a factor. (b) $4{,}095 = 3^2 \cdot 5 \cdot 7 \cdot 13$
5. (a) Divisible by 3; not divisible by 9. (b) The digit sum is a multiple of 3 but not of 9: it is 12.
6. (a) In 120 days. (b) On 3 days.
7. (a) 105 (b) $s(105) = 87$, so 105 is deficient.
8. (a) The digit sum is three times the digit. (b) 333, 666 and 999 (c) By 3: 3,333, 6,666 and 9,999. By 9: only 9,999.
9. (a) 2,730 = 2 · 3 · 5 · 7 · 13 (b) 13, 14 and 15 cartons
10. (a) $2 \cdot 3^3 \cdot 5$ (b) 5, 6, 9, 10, 15, 18, 27, 30, 45, 54 rows (c) 15 rows of 18, or 18 rows of 15
11. (a) 18 kits, each with 4 granola bars, 6 juice boxes and 7 bandannas. (b) 9 kits, each with 8 granola bars, 12 juice boxes and 15 bandannas.
12. (a) The GCF is a factor of the prime, whose only factors are 1 and itself; it is the prime exactly when the prime divides the other number. (b) GCF(23, 92) = 23 and GCF(29, 100) = 1.
13. (a) $1{,}188 = 2^2 \cdot 3^3 \cdot 11$ and $1{,}980 = 2^2 \cdot 3^2 \cdot 5 \cdot 11$. (b) GCF = 396, LCM = 5,940. (c) 396 · 5,940 = 2,352,240 = 1,188 · 1,980.

WORKED SOLUTIONS

1. Kevin writes the block 47 over and over to make longer and longer numbers: 47, then 4,747, and so on. (3 points)

(a) What is the fewest number of blocks that makes a number divisible by 3? Write that number.

(b) What is the fewest number of blocks that makes a number divisible by 9?

ANSWER

(a) 3 blocks: 474,747 (b) 9 blocks

SOLUTION

Each block adds 4 + 7 = 11 to the digit sum. So the digit sums are 11, 22, 33, 44, 55, 66, 77, 88, 99, and so on.

(a) The first of these divisible by 3 is 33 = 3 × 11, after 3 blocks. The number is 474,747. (11 and 22 leave remainders 2 and 1.)

(b) The first divisible by 9 is 99 = 9 × 11, after 9 blocks: 11, 22, 33, 44, 55, 66, 77 and 88 all leave a remainder when divided by 9. So it takes 9 blocks, an 18-digit number.

SCORING

1 point: the digit sum grows by 11 for each block (or the digit sums listed). 1 point: (a) 3 blocks, 474,747. 1 point: (b) 9 blocks, with the earlier digit sums ruled out.

2. Lily runs the sieve on a board from 1 to 200. (3 points)

(a) Andre writes 195 = 13 × 15 and says lantern 195 goes out in round 13. Is he right? Explain.

(b) Explain why a lantern that goes out in round 5 cannot be even and cannot be a multiple of 3.

(c) Which lanterns from 150 to 200 go out in round 5?

ANSWER

(a) No: 195 = 3 × 65, so it goes out in round 3. (b) An even lantern goes out in round 2 and a multiple of 3 in round 3, both before round 5, and a lantern goes out only once. (c) 155, 175 and 185

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SOLUTION

(a) A lantern goes out in the first round that reaches it, the round of its smallest prime factor. 195 is odd, but 195 = 3 × 65, so round 3 reaches it long before round 13. Andre is wrong: round 3.

(b) An even lantern (other than 2) is put out in round 2, and a multiple of 3 (other than 3) in round 3. Both rounds come before round 5, and a lantern that is out stays out, so round 5 can only put out lanterns that are neither even nor multiples of 3.

(c) The multiples of 5 from 150 to 200 are 150, 155, 160, ..., 200. The even ones went out in round 2, leaving 155, 165, 175, 185, 195. Of these, 165 = 3 × 55 and 195 = 3 × 65 went out in round 3. The rest, 155 = 5 × 31, 175 = 5 × 35 and 185 = 5 × 37, go out in round 5.

SCORING

1 point for "no, round 3" with 195 = 3 × 65 (or 3 as a factor) as the reason; 1 point for explaining that rounds 2 and 3 come first and a lantern goes out only once; 1 point for exactly 155, 175, 185.

3. (3 points)

(a) What is the smallest composite number that has no prime factor less than 11?

(b) What is the smallest composite number that has no prime factor less than 20?

(c) For part (a), explain how you know that no smaller composite number works.

ANSWER

(a) 121 = 11 × 11 (b) 529 = 23 × 23 (c) A composite number is $a \times b$ with a and b greater than 1; each has a prime factor of at least 11, so each is at least 11 and the product is at least 121.

SOLUTION

(a) The primes 11 and up are 11, 13, 17, ... A composite number is a product of two whole numbers greater than 1, and with no prime factor less than 11, each of those has all its prime factors 11 or more, so each is at least 11. The smallest such product is 11 × 11 = 121.

(b) The first prime that is 20 or more is 23 (19 is less than 20). In the same way the smallest product is 23 × 23 = 529.

(c) Write the composite number as $a \times b$ with a and b greater than 1. Each of a and b has a prime factor, and that prime is also a factor of the number, so it is at least 11. Then $a \geq 11$ and $b \geq 11$, so the number is at least 11 × 11 = 121, and 121 itself has no prime factor smaller than 11. This is why round 11 of the sieve starts at 121.

SCORING

1 point for 121; 1 point for 529; 1 point for explaining that both factors are at least 11, so the product is at least 11 × 11.

4. (3 points)

(a) Use the digit-sum test to show that 9 is a factor of 4,095.

(b) Find the prime factorization of 4,095, written with exponents, smallest prime first.

ANSWER

(a) 4 + 0 + 9 + 5 = 18, a multiple of 9, so 9 is a factor. (b) $4{,}095 = 3^2 \cdot 5 \cdot 7 \cdot 13$

SOLUTION

(a) The digits add to 4 + 0 + 9 + 5 = 18, and 18 is a multiple of 9, so 9 divides 4,095.

(b) 4,095 = 9 · 455 and 9 = 3 · 3. Then 455 ends in 5: 455 = 5 · 91, and 91 = 7 · 13, both prime. So $4{,}095 = 3^2 \cdot 5 \cdot 7 \cdot 13$. Check: 9 · 5 · 91 = 45 · 91 = 4,095.

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SCORING
1 point for (a), the digit sum 18 and the conclusion. 1 point for a tree or ladder ending in primes. 1 point for $3^2 \cdot 5 \cdot 7 \cdot 13$.

5. Lily's number is $2{,}352 = 2^4 \cdot 3 \cdot 7^2$. (3 points)

(a) Using only the prime factorization, decide whether 2,352 is divisible by 3, and whether it is divisible by 9.

(b) What do your answers to (a) tell you about the digit sum of 2,352? Check by adding the digits.

ANSWER
(a) Divisible by 3; not divisible by 9. (b) The digit sum is a multiple of 3 but not of 9: it is 12.

SOLUTION
(a) The factorization has a 3, so $2{,}352 = 3 \cdot (2^4 \cdot 7^2)$ and 3 divides it. $9 = 3^2$ needs two 3s and the factorization has only one; by uniqueness there is no other factorization with more, so 9 does not divide 2,352.

(b) A number is divisible by 3 exactly when its digit sum is, and by 9 exactly when its digit sum is. So the digit sum must be a multiple of 3 but not of 9. Check: 2 + 3 + 5 + 2 = 12, a multiple of 3 and not of 9.

SCORING
(a) 1 point: divisible by 3, from the single 3. 1 point: not divisible by 9, because 9 needs two 3s. (b) 1 point: digit sum is a multiple of 3 and not of 9, checked as 12.

6. A town's recycling truck collects glass every 6 days, paper every 8 days and metal every 10 days. All three were collected today. (3 points)

(a) In how many days are all three next collected on the same day?

(b) In the next 365 days, not counting today, on how many days are all three collected?

ANSWER
(a) In 120 days. (b) On 3 days.

SOLUTION
(a) The collection days are multiples of 6, 8 and 10, so the first shared one is LCM(6, 8, 10). List the multiples of 10 and test each against 6 and 8: a multiple of 10 that divides by 8 is 40, 80, 120, ..., and of these 40 and 80 do not divide by 6, while 120 = 6 × 20. All three are next collected in 120 days.

(b) The three collections line up at the multiples of 120: 120, 240 and 360. The next, 480, is past 365. That is 3 days.

SCORING
1 point for the least common multiple method with tests shown; 1 point for 120 days; 1 point for 3 days from the multiples of 120.

7. The two largest proper factors of a number are 35 and 21. (3 points)

(a) Find the number, and explain why no other number works.

(b) Is the number perfect, abundant or deficient? Show the sum.

ANSWER
(a) 105 (b) $s(105) = 87$, so 105 is deficient.

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SOLUTION

(a) The number has 35 and 21 as factors, so it is a common multiple of 35 and 21, and so a multiple of LCM(35, 21). Since 35 = 5 × 7 and 21 = 3 × 7, LCM(35, 21) = 3 × 5 × 7 = 105. So the number is 105, 210, 315, and so on.

Every one of these is a multiple of 3, so the number divided by 3 is one of its proper factors. That proper factor can be at most 35, so the number is at most 3 × 35 = 105. The number is 105.

Check: the factors of 105 are 1, 3, 5, 7, 15, 21, 35, 105, so its two largest proper factors are indeed 35 and 21.

(b) $s(105) = 1 + 3 + 5 + 7 + 15 + 21 + 35 = 87$, and 87 < 105, so 105 is deficient.

SCORING

1 point: the number is a multiple of LCM(35, 21) = 105. 1 point: explains why it cannot be larger than 105 (for example, the number divided by 3 would be a proper factor bigger than 35). 1 point: $s(105) = 87$, deficient.

8. Numbers such as 444 and 8,888 have all their digits the same. (4 points)

(a) Explain why every three-digit number with all its digits the same (111, 222, . . ., 999) is divisible by 3.

(b) Which three-digit numbers with all their digits the same are divisible by 9?

(c) Which four-digit numbers with all their digits the same are divisible by 3? Which are divisible by 9?

ANSWER

(a) The digit sum is three times the digit. (b) 333, 666 and 999 (c) By 3: 3,333, 6,666 and 9,999. By 9: only 9,999.

SOLUTION

(a) If every digit is d, the digit sum is $d + d + d = 3 \times d$, which is divisible by 3. So the number passes the test for 3.

(b) The digit sum $3 \times d$ must be divisible by 9. The possible digit sums are 3, 6, 9, ..., 27, and the multiples of 9 among them are 9, 18 and 27, for $d = 3$, 6 and 9. So 333, 666 and 999 (for example 333 = 9 × 37).

(c) The digit sum is $4 \times d$: one of 4, 8, 12, 16, 20, 24, 28, 32, 36. The multiples of 3 among them are 12, 24 and 36, for $d = 3$, 6 and 9: so 3,333, 6,666 and 9,999 are divisible by 3. The only multiple of 9 among them is 36, so only 9,999 is divisible by 9.

SCORING

(a) 1 point: the digit sum is 3 times the digit, so it is divisible by 3. (b) 1 point: 333, 666 and 999, and no others. (c) 1 point: 3,333, 6,666 and 9,999 for 3. 1 point: only 9,999 for 9, from the digit sum $4 \times d$.

9. Hiro stacks 2,730 identical cartons of crayons into one solid block, so many cartons long, so many wide and so many high. The three edge lengths are three whole numbers in a row (like 4, 5 and 6). (4 points)

(a) Find the prime factorization of 2,730.

(b) Use it to find the three edge lengths. Explain how the primes led you to them.

ANSWER

(a) 2,730 = 2 · 3 · 5 · 7 · 13 (b) 13, 14 and 15 cartons

SOLUTION

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(a) 2,730 = 10 · 273 and 10 = 2 · 5. The digits of 273 add to 12: 273 = 3 · 91, and 91 = 7 · 13. So 2,730 = 2 · 3 · 5 · 7 · 13.

(b) The edges multiply to 2,730, so the primes share out among the three edges, and the prime 13 must sit in one edge: that edge is 13 or a multiple of it such as 26. An edge of 26 or more is too big: the three edges would then all be at least 24, and 24 · 25 · 26 = 15,600 is already far more than 2,730. So one edge is exactly 13. Each prime appears only once, so no edge can be $12 = 2^2 \cdot 3$, which rules out 11, 12, 13 and 12, 13, 14. That leaves 13, 14 = 2 · 7 and 15 = 3 · 5, which use every prime exactly once. Check: 13 · 14 · 15 = 182 · 15 = 2,730.

SCORING

2 points for (a): 1 for a tree or ladder ending in primes, 1 for the factorization. 1 point for the edges 13, 14 and 15. 1 point for the reasoning: 13 must be one edge (or in one), and the other primes group as 2 · 7 and 3 · 5.

10. A marching band has 270 members. The director wants them in a rectangle of equal rows, with at least 4 rows and at least 4 members in each row. (4 points)

(a) Write the prime factorization of 270.

(b) Use it to list every possible number of rows.

(c) Which rectangle is closest to a square?

ANSWER

(a) $2 \cdot 3^3 \cdot 5$ (b) 5, 6, 9, 10, 15, 18, 27, 30, 45, 54 rows (c) 15 rows of 18, or 18 rows of 15

SOLUTION

(a) $270 = 27 \cdot 10 = 2 \cdot 3^3 \cdot 5$.

(b) The number of rows must be a factor of 270. A factor uses the 2 or not and the 5 or not (rows 1, 2, 5, 10 of a table) and none, one, two or three 3s (columns 1, 3, 9, 27):

Row 1: 1, 3, 9, 27. Row 2: 2, 6, 18, 54. Row 5: 5, 15, 45, 135. Row 10: 10, 30, 90, 270.

In order, the factors are

$$1, 2, 3, 5, 6, 9, 10, 15, 18, 27, 30, 45, 54, 90, 135, 270$$

At least 4 rows rules out 1, 2 and 3. At least 4 in each row means at most 270 ÷ 4 = 67.5 rows, which rules out 90, 135 and 270. That leaves 5, 6, 9, 10, 15, 18, 27, 30, 45, 54.

(c) The rows and the row length are closest when they are the two factors nearest each other: 15 · 18 = 270, and 16 and 17 do not divide 270. So 15 rows of 18 (or 18 rows of 15).

SCORING

(a) 1 point. (b) 1 point for all the factors from the factorization, 1 point for keeping exactly those from 4 to 67. (c) 1 point: 15 by 18.

11. A scout leader packs identical camp kits from 72 granola bars, 108 juice boxes and 126 bandannas, using all of them. (4 points)

(a) What is the largest number of kits she can pack, and what goes in each?

(b) Another box of 9 bandannas turns up, so she now has 135 bandannas. What is the largest number of identical kits she can pack now, using everything, and what goes in each?

ANSWER

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(a) 18 kits, each with 4 granola bars, 6 juice boxes and 7 bandannas. (b) 9 kits, each with 8 granola bars, 12 juice boxes and 15 bandannas.

SOLUTION

(a) The number of kits divides all three amounts. Work down the factors of 72: 72, 36 and 24 all fail on 126 (126 = 72 + 54, 126 = 3 × 36 + 18, 126 = 5 × 24 + 6). Next is 18: 72 = 18 × 4, 108 = 18 × 6, 126 = 18 × 7. So GCF(72, 108, 126) = 18: 18 kits of 4 granola bars, 6 juice boxes and 7 bandannas.

(b) 135 is odd, so no even number divides it, and every factor of 72 above 9 (72, 36, 24, 18, 12) is even. 9 works: the digit sums of 72, 108 and 135 are 9, 9 and 9, so 9 divides all three. So GCF(72, 108, 135) = 9: 9 kits of 8 granola bars, 12 juice boxes and 15 bandannas. Adding to one amount made the GCF smaller.

SCORING

(a) 1 point for 18 kits with a check that 18 divides all three amounts; 1 point for the contents. (b) 1 point for 9 kits, with a reason no larger factor of 72 works (for example, 135 is odd); 1 point for the contents.

12. (4 points)

(a) Take a prime number and any whole number other than 0. Explain why the GCF of the two numbers can only be 1 or the prime itself, and say when it is the prime.

(b) Use this to find GCF(23, 92) and GCF(29, 100).

ANSWER

(a) The GCF is a factor of the prime, whose only factors are 1 and itself; it is the prime exactly when the prime divides the other number. (b) GCF(23, 92) = 23 and GCF(29, 100) = 1.

SOLUTION

(a) The GCF is a common factor, so it is on the prime's factor list. A prime has exactly two factors, 1 and itself, so the GCF is one of them. It is the prime when the prime is also a factor of the other number, that is when the prime divides it with nothing left over; otherwise only 1 is common and the GCF is 1.

(b) 92 = 23 × 4, so 23 divides 92 and GCF(23, 92) = 23. 100 = 29 × 3 + 13, so 29 does not divide 100 and GCF(29, 100) = 1.

SCORING

(a) 1 point for saying the GCF must be a factor of the prime, so 1 or the prime; 1 point for "it is the prime exactly when the prime divides the other number". (b) 1 point for each correct GCF, with the division that decides it.

13. (4 points)

(a) Write 1,188 and 1,980 as products of primes.

(b) Find the GCF and the LCM of 1,188 and 1,980.

(c) Check your answers to (b) with the product rule.

ANSWER

(a) $1{,}188 = 2^2 \cdot 3^3 \cdot 11$ and $1{,}980 = 2^2 \cdot 3^2 \cdot 5 \cdot 11$. (b) GCF = 396, LCM = 5,940. (c) 396 · 5,940 = 2,352,240 = 1,188 · 1,980.

SOLUTION

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(a) Both numbers are even, and both digit sums are 18, so both are divisible by 9. Dividing: 1,188 = 9 · 132 = 9 · 4 · 33 and 1,980 = 9 · 220 = 9 · 4 · 55, so

$$1{,}188 = 2^2 \cdot 3^3 \cdot 11 \qquad 1{,}980 = 2^2 \cdot 3^2 \cdot 5 \cdot 11$$

(b) Shared primes 2, 3 and 11, at the smaller powers 2^2, 3^2 and 11^1; the 5 is only in 1,980.

$$\text{GCF} = 2^2 \cdot 3^2 \cdot 11 = 396$$

Greater powers of every prime:

$$\text{LCM} = 2^2 \cdot 3^3 \cdot 5 \cdot 11 = 4 \cdot 27 \cdot 55 = 5{,}940$$

(c) 1,188 · 1,980 = 2,352,240 and 396 · 5,940 = 2,352,240. They agree.

SCORING

1 point: 1,188 factorized correctly. 1 point: 1,980 factorized correctly. 1 point: GCF 396 and LCM 5,940. 1 point: both products computed and compared.

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