Math 6 Unit 1 worksheet: Perfect, abundant, deficient and amicable numbers (answer key)

ANSWERS AT A GLANCE

1. (a) $s(26) = 16$, deficient (b) $s(30) = 42$, abundant (c) $s(44) = 40$, deficient (d) $s(55) = 17$, deficient (e) $s(60) = 108$, abundant (f) $s(63) = 41$, deficient
2. $s(76) = 64$, so 76 is deficient: the sum is 12 less than 76.
3. (a) He added 32 itself, which is not a proper factor. $s(32) = 31 < 32$, so 32 is deficient. (b) $1024 = 2^{10}$
4. (a) In an amicable pair the proper factors of 66 add to its partner, so the partner is $s(66)$. (b) $s(66) = 78$, but $s(78) = 90 \neq 66$, so 66 is not in an amicable pair.
5. (a) 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120 (b) $s(120) = 240$, so 120 is abundant; the sum is exactly twice 120.
6. If p is prime, the proper factors of $p \times p$ are only 1 and p, so $s(p \times p) = 1 + p$, and $1 + p$ is less than $p \times p$ because $p \times p \geq p + p$.
7. (a) 90 = 15 × 6, abundant. (b) $s(98) = 73$, deficient. (c) 140 = 5 × 28 (also 7 × 20), abundant. (d) $s(175) = 73$, deficient.

WORKED SOLUTIONS

1. Find $s(n)$, the sum of the proper factors. Then say whether the number is perfect, abundant or deficient.

(a) 26 (b) 30 (c) 44

(d) 55 (e) 60 (f) 63

ANSWER

(a) $s(26) = 16$, deficient (b) $s(30) = 42$, abundant (c) $s(44) = 40$, deficient (d) $s(55) = 17$, deficient (e) $s(60) = 108$, abundant (f) $s(63) = 41$, deficient

SOLUTION

List the factors in pairs that multiply to the number, then add all but the number itself.

(a) 26 = 1 × 26 = 2 × 13. Proper factors 1, 2, 13: $s(26) = 16 < 26$, deficient.

(b) 30 = 1 × 30 = 2 × 15 = 3 × 10 = 5 × 6. Proper factors 1, 2, 3, 5, 6, 10, 15: $s(30) = 42 > 30$, abundant.

(c) 44 = 1 × 44 = 2 × 22 = 4 × 11. Proper factors 1, 2, 4, 11, 22: $s(44) = 40 < 44$, deficient.

(d) 55 = 1 × 55 = 5 × 11. Proper factors 1, 5, 11: $s(55) = 17 < 55$, deficient.

(e) 60 = 1 × 60 = 2 × 30 = 3 × 20 = 4 × 15 = 5 × 12 = 6 × 10. Proper factors 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30: $s(60) = 108 > 60$, abundant.

(f) 63 = 1 × 63 = 3 × 21 = 7 × 9. Proper factors 1, 3, 7, 9, 21: $s(63) = 41 < 63$, deficient.

2. $76 = 2^2 \times 19$. Use the factorization to list the proper factors of 76 and find $s(76)$. Is 76 perfect, abundant or deficient, and by how much does the sum miss 76?

ANSWER

$s(76) = 64$, so 76 is deficient: the sum is 12 less than 76.

SOLUTION

By unique factorization every factor of 76 is a power of 2 (up to the one in $2^2 \times 19$), or such a power times the odd prime. Leaving out 76 itself, the proper factors are

$$1, 2, 4, 19, 38$$

and they add to $s(76) = 64$. The difference between 64 and 76 is 12, and the sum is less than the number, so 76 is deficient.

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3. Daniel adds up the factors of 32 and gets 1 + 2 + 4 + 8 + 16 + 32 = 63. Since 63 > 32, he says 32 is abundant.

(a) What did Daniel do wrong? Is 32 perfect, abundant or deficient?

(b) The proper factors of a power of 2 add to 1023. Which power of 2 is it? Give it as a number and in the form 2^k.

ANSWER

(a) He added 32 itself, which is not a proper factor. $s(32) = 31 < 32$, so 32 is deficient. (b) $1024 = 2^{10}$

SOLUTION

(a) A proper factor of a number is a factor other than the number itself, so 32 must be left out. The proper factors are 1, 2, 4, 8, 16, and

$$s(32) = 1 + 2 + 4 + 8 + 16 = 31$$

Since 31 < 32, the number 32 is deficient (not abundant).

(b) Powers of 2 fall short by exactly one: $s(2^k) = 2^k - 1$. So the power of 2 is one more than the sum, 1023 + 1 = 1024. Doubling from 2: 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, which is ten doublings, so $1024 = 2^{10}$. Check: its proper factors 1, 2, 4, ..., 512 add to 1023.

4. (a) Suppose 66 belongs to an amicable pair. Explain why its partner could only be the number $s(66)$.

(b) Find $s(66)$ and use it to decide whether 66 belongs to an amicable pair.

ANSWER

(a) In an amicable pair the proper factors of 66 add to its partner, so the partner is $s(66)$. (b) $s(66) = 78$, but $s(78) = 90 \neq 66$, so 66 is not in an amicable pair.

SOLUTION

(a) If 66 and b are an amicable pair, the definition says $s(66) = b$. So b is not free to choose: the only possible partner is the one number $s(66)$.

(b) 66 = 1 × 66 = 2 × 33 = 3 × 22 = 6 × 11, so

$$s(66) = 1 + 2 + 3 + 6 + 11 + 22 + 33 = 78$$

The only candidate is 78, and 78 ≠ 66. Now check the other direction: 78 = 1 × 78 = 2 × 39 = 3 × 26 = 6 × 13, so

$$s(78) = 1 + 2 + 3 + 6 + 13 + 26 + 39 = 90$$

Since 90 ≠ 66, the sum from 78 does not lead back. The one possible partner fails, so 66 is not in any amicable pair.

5. Euclid's rule with $k = 4$ would give $2^3 \times (2^4 - 1) = 8 \times 15 = 120$, but 15 is not prime, so the rule says nothing about 120.

(a) Use $120 = 2^3 \times 3 \times 5$ to list all 16 factors of 120.

(b) Find $s(120)$. Is 120 perfect, abundant or deficient? How does $s(120)$ compare with 120?

ANSWER

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(a) 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120 (b) $s(120) = 240$, so 120 is abundant; the sum is exactly twice 120.

SOLUTION

(a) Every factor is a power of 2 up to 2^3 times one of 1, 3, 5, 15 (the factors of 3 × 5):

	1	3	5	15
1	1	3	5	15
2	2	6	10	30
4	4	12	20	60
8	8	24	40	120

In order: 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120.

(b) Add each row: 1 + 3 + 5 + 15 = 24, then 48, 96 and 192 (each row is double the one above). All sixteen add to 24 + 48 + 96 + 192 = 360. Leave out 120:

$$s(120) = 360 - 120 = 240$$

240 > 120, so 120 is abundant, and 240 = 2 × 120: the proper factors add to exactly twice the number. (It had to be abundant anyway, since 120 = 20 × 6 is a multiple of the perfect number 6.)

6. Take any prime and square it, for example $7^2 = 49$ or $13^2 = 169$. Explain why the square of a prime is always deficient.

ANSWER

If p is prime, the proper factors of $p \times p$ are only 1 and p, so $s(p \times p) = 1 + p$, and $1 + p$ is less than $p \times p$ because $p \times p \geq p + p$.

SOLUTION

Call the prime p. By unique factorization a factor of $p \times p$ can contain no prime but p, and at most two of them, so the factors are 1, p and $p \times p$. Leaving out the number itself, the proper factors are 1 and p:

$$s(p \times p) = 1 + p$$

Since p is at least 2, $p \times p$ is at least $2 \times p = p + p$, and $p + p$ is more than $p + 1$ because $p > 1$. So $s(p \times p) < p \times p$, and the square is deficient. For example $s(49) = 1 + 7 = 8 < 49$ and $s(169) = 1 + 13 = 14 < 169$.

7. You know that 6 and 28 are perfect, and that 12 and 20 are abundant. For each number below, either show it is a multiple of one of those four (which proves it is abundant), or find s of the number and say whether it is perfect, abundant or deficient.

(a) 90 (b) 98 (c) 140 (d) 175

ANSWER

(a) 90 = 15 × 6, abundant. (b) $s(98) = 73$, deficient. (c) 140 = 5 × 28 (also 7 × 20), abundant. (d) $s(175) = 73$, deficient.

SOLUTION

Every multiple (other than the number itself) of a perfect or abundant number is abundant.

(a) 90 = 15 × 6 and 6 is perfect, so 90 is abundant.

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(b) 98 is not a multiple of 6, 12, 20 or 28 (98 ÷ 28 leaves 14), so the rule does not help. $98 = 2 \times 7^2$, so the factors are 1, 2, 7, 14, 49, 98 and

$$s(98) = 1 + 2 + 7 + 14 + 49 = 73$$

73 < 98, so 98 is deficient.

(c) 140 = 5 × 28 and 28 is perfect, so 140 is abundant. (140 = 7 × 20 works too.)

(d) 175 is odd, so it is not a multiple of any of the four even numbers. $175 = 5^2 \times 7$, so the factors are 1, 5, 7, 25, 35, 175 and

$$s(175) = 1 + 5 + 7 + 25 + 35 = 73$$

73 < 175, so 175 is deficient.

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