Math 8 Unit 1 test: Exponents and scientific notation (answer key)

13 questions, 43 points

ANSWERS AT A GLANCE

1. (a) $(x+2)^7$ (b) $3^7 = 2187$
2. (a) 2^{24} (b) Larger: $16^3 \cdot 8^4$ is 4 times 4^{11}.
3. (a) $2^{-5} = \frac{1}{32}$ square meter (b) 16 cards (c) A4
4. (a) $x = -1$ and $x = 2$ (b) $x - 2$
5. (a) $6x^{n+3}$ (b) $n = 7$
6. (a) $0.03125 = 3.125 \times 10^{-2}$ (b) $0.0016 = 1.6 \times 10^{-3}$ (c) $0.000125 = 1.25 \times 10^{-4}$
7. (a) 1.6×10^{-11} square meters. (b) About 2×10^7 pixels.
8. (a) 3.0×10^{-4} kg (b) 3.0×10^{-1} g
9. (a) 5^{2n}; the m in the exponent cancels. (b) $n = 3$
10. (a) $9a^4b^2$ square centimeters. (b) $54a^7b^5$ cubic centimeters. (c) $4a^3b^3$ centimeters deep, which is $\frac{2}{3}$ of the height.
11. $(2 \times 10^3)^2 = 4 \times 10^6$, $\frac{10^8}{4 \times 10^2} = 2.5 \times 10^5$, $5^{-1} \times 10^7 = 2 \times 10^6$, $3 \times 10^{-2} \times 10^8 = 3 \times 10^6$. Least to greatest: $2.5 \times 10^5 < 2 \times 10^6 < 3 \times 10^6 < 4 \times 10^6$.
12. $9a^2$ stones
13. (a) $0.001 \le N < 0.01$ (b) No: $0.00099 = 9.9 \times 10^{-4}$.

WORKED SOLUTIONS

1. Here x is a number other than −2. (3 points)

(a) Write $\dfrac{(x+2)^9 \cdot (x+2)^4}{(x+2)^6}$ as a single power of $(x+2)$. Explain why the rules apply even though the base is not a single number or letter.

(b) Find the value of the expression when $x = 1$.

ANSWER

(a) $(x+2)^7$ (b) $3^7 = 2187$

SOLUTION

(a) For any one value of x, the base $x + 2$ is a single number, and it is not 0 because $x \neq -2$. The rules hold for any number as the base (any nonzero number for a quotient), so they apply here. The top is a product of powers of $x + 2$, so the exponents add: 9 + 4 = 13. Then 13 > 6, so the exponents subtract:

$$\frac{(x+2)^9 \cdot (x+2)^4}{(x+2)^6} = \frac{(x+2)^{13}}{(x+2)^6} = (x+2)^7$$

(b) At $x = 1$ the base is 1 + 2 = 3, so the value is $3^7 = 2187$.

SCORING

1 point for $(x+2)^{13}$ on top with the reason that $x + 2$ is one number; 1 point for $(x+2)^7$; 1 point for $3^7 = 2187$.

2. (3 points)

(a) Write $16^3 \cdot 8^4$ as a single power of 2.

(b) Is $16^3 \cdot 8^4$ larger or smaller than 4^{11}, and how many times as large? Do not work out any of the numbers in full.

ANSWER

(a) 2^{24} (b) Larger: $16^3 \cdot 8^4$ is 4 times 4^{11}.

SOLUTION

Free sample from math.wiki for classroom use. Do not post. math.wiki

(a) Rewrite each base as a power of 2, multiply exponents, then add them:

$$16^3 \cdot 8^4 = (2^4)^3 \cdot (2^3)^4 = 2^{12} \cdot 2^{12} = 2^{24}$$

(b) $4^{11} = (2^2)^{11} = 2^{22}$. Since $2^{24} = 2^{22} \cdot 2^2$, the product is larger, and it is $2^2 = 4$ times 4^{11}.

SCORING

(a) 1 point for rewriting the bases as powers of 2, 1 point for 2^{24}. (b) 1 point for $4^{11} = 2^{22}$ and the conclusion "larger, 4 times".

3. In the A series of paper sizes, an A0 sheet has an area of 1 square meter and each size is half the one before, so an An sheet has an area of 2^{-n} square meters. (A4 is the usual printer paper.) (3 points)

(a) Write the area of an A5 sheet as a power of 2 and as a fraction of a square meter.

(b) How many A7 cards can be cut from one A3 sheet with nothing wasted? Use the quotient rule.

(c) Two A5 sheets laid side by side cover the same area as one sheet of which size? Use the product rule.

ANSWER

(a) $2^{-5} = \dfrac{1}{32}$ square meter (b) 16 cards (c) A4

SOLUTION

(a) An A5 sheet has area $2^{-5} = \dfrac{1}{2^5} = \dfrac{1}{32}$ square meter.

(b) Divide the areas, subtracting exponents:

$$\frac{2^{-3}}{2^{-7}} = 2^{-3-(-7)} = 2^4 = 16$$

(c) The two sheets cover $2 \cdot 2^{-5} = 2^1 \cdot 2^{-5} = 2^{1+(-5)} = 2^{-4}$ square meters, the area of an A4 sheet.

SCORING

1 point for each part: (a) 2^{-5} and $\dfrac{1}{32}$; (b) 16 from a correct quotient of powers; (c) 2^{-4}, so A4.

4. Consider the expression $\dfrac{(x+1)^0}{(x-2)^{-1}}$. (3 points)

(a) For which values of x does the expression have no value? Explain each.

(b) Simplify the expression for every other x.

ANSWER

(a) $x = -1$ and $x = 2$ (b) $x - 2$

SOLUTION

(a) The top $(x+1)^0$ equals 1 only when its base is nonzero; at $x = -1$ it is 0^0, which has no value. The bottom $(x-2)^{-1} = \dfrac{1}{x-2}$ needs $x - 2 \neq 0$; at $x = 2$ it would be $0^{-1} = \dfrac{1}{0}$. So $x = -1$ and $x = 2$ are left out.

(b) For every other x, the top is 1 and the bottom is $\dfrac{1}{x-2}$, so

$$\frac{1}{\frac{1}{x-2}} = x - 2$$

Free sample from math.wiki for classroom use. Do not post. math.wiki

(The reciprocal of a reciprocal is the number you started from.) For example at $x = 5$: $\dfrac{6^0}{3^{-1}} = \dfrac{1}{\frac{1}{3}} = 3 = 5 - 2$.

SCORING

1 point for $x = -1$ with the reason (0^0); 1 point for $x = 2$ with the reason (division by zero); 1 point for $x - 2$.

5. Here n is a positive whole number and $x \neq 0$. (3 points)

(a) Simplify $\dfrac{(3x^n)^2 \cdot 4x^{n+3}}{6x^{2n}}$.

(b) For which n is the expression equal to $6x^{10}$ for every nonzero x?

ANSWER

(a) $6x^{n+3}$ (b) $n = 7$

SOLUTION

(a) The power of the parentheses first: $(3x^n)^2 = 9x^{2n}$. The product on top: $9x^{2n} \cdot 4x^{n+3} = 36x^{2n+n+3} = 36x^{3n+3}$. Then divide: $\dfrac{36}{6} = 6$ and $(3n + 3) - 2n = n + 3$, so

$$\frac{(3x^n)^2 \cdot 4x^{n+3}}{6x^{2n}} = 6x^{n+3}$$

(b) The exponents must agree: $n + 3 = 10$, so $n = 7$.

Check (a) at $n = 1$, $x = 2$: the left side is $\dfrac{36 \cdot 4 \cdot 16}{6 \cdot 4} = 96$, and $6 \cdot 2^4 = 96$.

SCORING

1 point for $(3x^n)^2 = 9x^{2n}$ and the top as $36x^{3n+3}$. 1 point for $6x^{n+3}$. 1 point for $n = 7$.

6. Write each power as a decimal, and then in scientific notation.

(a) 2^{-5} (b) 5^{-4} (c) 20^{-3} (3 points)

ANSWER

(a) $0.03125 = 3.125 \times 10^{-2}$ (b) $0.0016 = 1.6 \times 10^{-3}$ (c) $0.000125 = 1.25 \times 10^{-4}$

SOLUTION

A negative exponent means one over the positive power.

(a) $2^{-5} = \dfrac{1}{2^5} = \dfrac{1}{32} = 0.03125$. The 3 is in the second decimal place: 3.125×10^{-2}.

(b) $5^{-4} = \dfrac{1}{5^4} = \dfrac{1}{625} = 0.0016$. The 1 is in the third decimal place: 1.6×10^{-3}.

(c) $20^{-3} = \dfrac{1}{20^3} = \dfrac{1}{8000} = 0.000125$. The 1 is in the fourth decimal place: 1.25×10^{-4}.

SCORING

1 point for each part: the correct decimal and the correct scientific notation. Take off $\dfrac{1}{2}$ point for a correct decimal with a wrong exponent.

7. A camera sensor is covered with tiny square pixels, each 4×10^{-6} meters on a side. (3 points)

(a) Find the area of one pixel, in square meters, in scientific notation.

Free sample from math.wiki for classroom use. Do not post. math.wiki

(b) The sensor has an area of 3.2×10^{-4} square meters and is covered by pixels with no gaps. About how many pixels does it have?

ANSWER
(a) 1.6×10^{-11} square meters. (b) About 2×10^{7} pixels.

SOLUTION
(a) The area of a square is its side squared, and a power of a product is the product of the powers:

$$\begin{aligned}(4 \times 10^{-6})^2 &= 4^2 \times (10^{-6})^2 \\ &= 16 \times 10^{-12} \\ &= 1.6 \times 10^{-11}\ \text{m}^2\end{aligned}$$

(b) The number of pixels is the sensor's area divided by the area of one pixel:

$$\begin{aligned}\frac{3.2 \times 10^{-4}}{1.6 \times 10^{-11}} &= 2 \times 10^{-4-(-11)} \\ &= 2 \times 10^{7}\end{aligned}$$

That is about twenty million pixels.

SCORING
1 point for squaring both parts in (a), giving 16×10^{-12}. 1 point for renormalizing to 1.6×10^{-11}. 1 point for (b), 2×10^{7}, set up as a division.

8. A spool of fishing line holds 1.50×10^{2} m of line. The line alone has a mass of 4.5×10^{-2} kg. (3 points)

(a) Find the mass of one meter of the line in kilograms, in scientific notation, to an honest number of significant figures.

(b) There are 10^{3} grams in a kilogram. Write the mass of one meter in grams, in scientific notation, keeping its significant figures.

ANSWER
(a) 3.0×10^{-4} kg (b) 3.0×10^{-1} g

SOLUTION
(a) Divide the leading numbers and subtract the exponents:

$$\frac{4.5 \times 10^{-2}}{1.50 \times 10^{2}} = \frac{4.5}{1.50} \times 10^{-2-2} = 3 \times 10^{-4}$$

The mass has two significant figures and the length three, so the quotient keeps two. The second figure is a zero and must be written: 3.0×10^{-4} kg.

(b) The factor 10^{3} comes from a unit definition, so it is exact: $3.0 \times 10^{-4} \times 10^{3} = 3.0 \times 10^{-1}$ g, still two figures (as a decimal, 0.30 g).

SCORING
(a) 1 point for 3×10^{-4} from the exponent rules, 1 point for two significant figures written as 3.0. (b) 1 point for 3.0×10^{-1} g or 0.30 g (a plain 3×10^{-1} or 0.3 earns 0).

9. Here m and n are positive whole numbers with $m > n$. (4 points)

Free sample from math.wiki for classroom use. Do not post. math.wiki

(a) Write $\dfrac{5^m \cdot 5^n}{5^{m-n}}$ as a single power of 5, and explain why its value does not depend on m.

(b) For which n does the expression equal 15,625?

ANSWER

(a) 5^{2n}; the m in the exponent cancels. (b) $n = 3$

SOLUTION

(a) The top is a product of powers of 5, so $5^m \cdot 5^n = 5^{m+n}$. The bottom has exponent $m - n$, a positive whole number because $m > n$, and $m + n > m - n$ because $n > 0$. So the quotient of powers applies:

$$\frac{5^{m+n}}{5^{m-n}} = 5^{(m+n)-(m-n)} = 5^{2n}$$

The m factors that the top gains from 5^m are matched by m factors on the bottom, so m drops out and only n is left in the exponent.

(b) $15{,}625 = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 5^6$. So $5^{2n} = 5^6$, which gives $2n = 6$ and $n = 3$. Any $m > 3$ works; for example $m = 4$ gives $\dfrac{5^4 \cdot 5^3}{5^1} = 5^6$.

SCORING

1 point for $5^m \cdot 5^n = 5^{m+n}$; 1 point for subtracting to 5^{2n}; 1 point for explaining that m cancels; 1 point for $n = 3$ with $15{,}625 = 5^6$.

10. A fish tank is a box whose base is a square with sides $3a^2b$ centimeters long. The tank is $6a^3b^3$ centimeters tall, and a and b are positive. (4 points)

(a) Write the area of the base as a monomial.

(b) Write the volume of the tank as a monomial.

(c) The water in the tank has a volume of $36a^7b^5$ cubic centimeters. How deep is the water, and what fraction of the tank's height does it fill?

ANSWER

(a) $9a^4b^2$ square centimeters. (b) $54a^7b^5$ cubic centimeters. (c) $4a^3b^3$ centimeters deep, which is $\dfrac{2}{3}$ of the height.

SOLUTION

(a) The base is a square, so its area is the side squared, coefficient included: $(3a^2b)^2 = 3^2a^{2\cdot2}b^2 = 9a^4b^2$ square centimeters.

(b) Volume is base area times height: $(9a^4b^2)(6a^3b^3) = 54a^{4+3}b^{2+3} = 54a^7b^5$ cubic centimeters.

(c) The water is a box on the same base, so its depth is its volume divided by the base area:

$$\frac{36a^7b^5}{9a^4b^2} = 4a^{7-4}b^{5-2} = 4a^3b^3$$

As a fraction of the height, $\dfrac{4a^3b^3}{6a^3b^3} = \dfrac{2}{3}$, since the equal powers cancel. Check: the water's volume is $\dfrac{36}{54} = \dfrac{2}{3}$ of the tank's.

SCORING

Free sample from math.wiki for classroom use. Do not post. math.wiki

(a) 1 point for $9a^4b^2$ (the coefficient must be squared). (b) 1 point for $54a^7b^5$. (c) 1 point for the depth $4a^3b^3$, found by dividing the water's volume by the base area; 1 point for $\frac{2}{3}$.

11. Write each number in scientific notation. Then list the four numbers from least to greatest.

$$(2 \times 10^3)^2, \qquad \frac{10^8}{4 \times 10^2}, \qquad 5^{-1} \times 10^7, \qquad 3 \times 10^{-2} \times 10^8$$

(4 points)

ANSWER

$(2 \times 10^3)^2 = 4 \times 10^6$, $\frac{10^8}{4 \times 10^2} = 2.5 \times 10^5$, $5^{-1} \times 10^7 = 2 \times 10^6$, $3 \times 10^{-2} \times 10^8 = 3 \times 10^6$. Least to greatest: $2.5 \times 10^5 < 2 \times 10^6 < 3 \times 10^6 < 4 \times 10^6$.

SOLUTION

- A product raised to a power: $(2 \times 10^3)^2 = 2^2 \times 10^6 = 4 \times 10^6$.
- Dividing powers of ten: $\frac{10^8}{4 \times 10^2} = \frac{1}{4} \times 10^6 = 0.25 \times 10^6 = 2.5 \times 10^5$.
- A negative exponent: $5^{-1} = \frac{1}{5} = 0.2$, so $5^{-1} \times 10^7 = 0.2 \times 10^7 = 2 \times 10^6$.
- Multiplying powers of ten: $3 \times 10^{-2} \times 10^8 = 3 \times 10^6$.

Sort by exponent: only 2.5×10^5 has exponent 5, so it is the least. The other three have exponent 6, and their leading parts give 2 < 3 < 4:

$$2.5 \times 10^5 < 2 \times 10^6 < 3 \times 10^6 < 4 \times 10^6$$

That is $\frac{10^8}{4 \times 10^2} < 5^{-1} \times 10^7 < 3 \times 10^{-2} \times 10^8 < (2 \times 10^3)^2$.

SCORING

1 point for 4×10^6 and 3×10^6; 1 point for 2.5×10^5 (leading part rewritten from 0.25); 1 point for 2×10^6; 1 point for the correct order, following from the student's values.

12. A square courtyard has sides $6a^3$ meters long, where a is a positive whole number. It is paved with square stones whose sides are $2a^2$ meters long, with no gaps and no overlaps. How many stones cover the courtyard? Give your answer as a monomial in a. (3 points)

ANSWER

$9a^2$ stones

SOLUTION

The area of a square is its side squared, and each side is a product, so the square goes to every factor:

$$(6a^3)^2 = 6^2(a^3)^2 = 36a^6 \text{ m}^2$$
$$(2a^2)^2 = 2^2(a^2)^2 = 4a^4 \text{ m}^2$$

The number of stones is the courtyard's area divided by one stone's area:

Free sample from math.wiki for classroom use. Do not post. math.wiki

$$\frac{36a^6}{4a^4} = 9a^{6-4} = 9a^2$$

A check: along one side fit $\frac{6a^3}{2a^2} = 3a$ stones, and $3a$ rows of $3a$ stones make $(3a)^2 = 9a^2$.

SCORING
1 point for $36a^6$; 1 point for $4a^4$; 1 point for dividing to get $9a^2$.

13. A positive number N has exponent -3 when it is written in scientific notation. (4 points)

(a) Fill in the blanks with decimals: _____ $\leq N <$ _____. Explain how you know.

(b) Owen says 0.00099 has exponent -3, "because it has three zeros after the decimal point." Is he right? Explain, and write 0.00099 in scientific notation.

ANSWER
(a) $0.001 \leq N < 0.01$ (b) No: $0.00099 = 9.9 \times 10^{-4}$.

SOLUTION
(a) $N = a \times 10^{-3}$ with $1 \leq a < 10$. Multiply all three parts by 10^{-3}: $1 \times 10^{-3} \leq N < 10 \times 10^{-3} = 10^{-2}$. As decimals, $0.001 \leq N < 0.01$. The smallest such number is 0.001 itself; every one is less than 0.01.

(b) Owen is wrong. The exponent is minus the decimal place of the first digit that is not zero, not minus the count of zeros. In 0.00099 the first 9 is in the fourth decimal place, so $0.00099 = 9.9 \times 10^{-4}$. This also agrees with (a): 0.00099 is less than 0.001, so its exponent cannot be -3.

SCORING
(a) 2 points: 1 point for 0.001 and 0.01 in the right places, 1 point for a reason (the leading number lies from 1 up to but not including 10). (b) 2 points: 1 point for saying Owen is wrong with a reason (the place of the first digit that is not zero, or 0.00099 < 0.001), 1 point for 9.9×10^{-4}.

Free sample from math.wiki for classroom use. Do not post. math.wiki