Concavity and changing rates

Increasing or decreasing tells you the sign of a rate. Is the rate itself growing or shrinking, and how does that show in the way a graph bends?

WORDS TO KNOW

Secant slope $m(a, b)$: the slope $\frac{f(b)-f(a)}{b-a}$ of the line from $x = a$ to $x = b$ on the graph.

Concave down: the secant slopes decrease as you move right. The graph bends like a cap.

First and second differences: in a table, the changes between outputs, and then the changes between those changes.

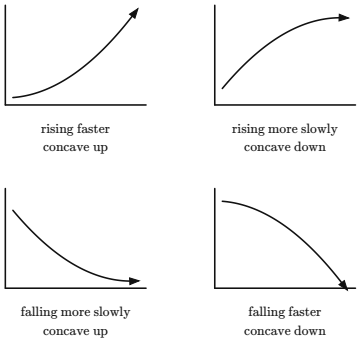
Concave up: the secant slopes increase as you move right. The graph bends like a cup.

Point of inflection: a point on the graph where the concavity changes from up to down or from down to up.

KEY IDEAS

1. Concavity is about the rate, not the direction

Concave up means the rate of change increases from left to right. Concave down means it decreases. This is separate from rising and falling: the sign of the rate says up or down, and whether the rate grows says which way the graph bends. A falling graph can be concave up when it falls less and less steeply.



For $a < b < c$ in an interval:

concave up means $m(a, b) < m(b, c)$; concave down means $m(a, b) > m(b, c)$.

2. Read concavity from a table

If the inputs are equally spaced, positive second differences mean concave up, and negative second differences mean concave down.

If the spacing is not equal, divide each change by its step to get rates, and compare the rates.

3. Shortcuts for common functions

Quadratic $px^2 + qx + c$: concave up everywhere if $p > 0$, concave down everywhere if $p < 0$. No point of inflection.

Cubic $ax^3 + bx^2 + cx + e$: one point of inflection, at $x = -\frac{b}{3a}$. Its rate of change at x is $3ax^2 + 2bx + c$. If $a > 0$, it is concave down on the left and concave up on the right; if $a < 0$, the reverse.

Exponential ab^x with $a > 0$: concave up everywhere, whether $b > 1$ or $0 < b < 1$.

Line: neither, since all its secant slopes are equal.

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WATCH OUT

- **Concave up does not mean rising.** It means the rate is increasing. A rate going from -5 to -2 is increasing.
- **Flat is not inflection.** $y = x^4$ is flat at the origin but concave up on both sides. A turning point, where the graph switches between rising and falling, is also a different thing.
- **An inflection point must be on the graph.** $y = \frac{1}{x}$ changes concavity across $x = 0$, but 0 is not in its domain, so there is no point of inflection.
- **A table only samples.** Its differences describe the listed points; the function may bend another way between them.

WORKED EXAMPLES

Example 1. A leaking tank holds the liters of water below, read every 2 hours. Describe how the amount changes.

hours	0	2	4	6	8
liters	80	76	68	56	40

1. First differences: -4, -8, -12, -16. Divide by the 2-hour step: the rates are -2, -4, -6 and -8 liters per hour.
2. Every rate is negative, so the amount is decreasing.
3. The rates decrease (the second differences are all -4), so the table is concave down.

Answer: decreasing and concave down: the tank loses water faster and faster.

Example 2. Find where $f(x) = -2x^3 + 12x^2 - 10x + 3$ is concave up and concave down, and its point of inflection.

1. Here $a = -2$ and $b = 12$, so the point of inflection is at $x = -12 \div (3 \cdot (-2)) = 2$.
2. The rate of change is $-6x^2 + 24x - 10 = -6(x - 2)^2 + 14$. It increases up to $x = 2$ and decreases after.
3. So f is concave up on $(-\infty, 2)$ and concave down on $(2, \infty)$, as the rule for $a < 0$ says.
4. $f(2) = -16 + 48 - 20 + 3 = 15$.

Answer: concave up on $(-\infty, 2)$, concave down on $(2, \infty)$, point of inflection $(2, 15)$. There the rate is largest, 14.

Example 3. Show that $g(x) = 8(0.5)^x$ is concave up, even though it falls.

1. Values at $x = 0, 1, 2, 3$: 8, 4, 2, 1.
2. First differences: -4, -2, -1. Each is half the one before.
3. They are negative, so g falls. But they increase toward 0, so the graph is concave up.

Answer: g is decreasing and concave up: it falls by less each step. This fits the exponential rule, with $a = 8 > 0$.

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