Degree and the shape of a polynomial graph

Without plotting a single point, what can the formula of a polynomial tell you about the shape of its graph?

WORDS TO KNOW

Degree: the highest power of x in the polynomial, after like terms are collected.

Leading coefficient: the number in front of the leading term.

Turning point: a peak or a valley, where the graph switches between rising and falling.

Leading term: the term with the highest power, wherever it is written. In $8x - x^2 + 2x^3$ it is $2x^3$.

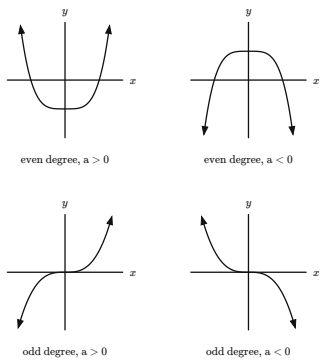
Real zero: a number r with $p(r) = 0$. It is an x-intercept of the graph.

End behavior: what the graph does far to the left and far to the right. "As $x \to \infty$, $p(x) \to -\infty$" says that far to the right the graph falls without end.

KEY IDEAS

1. The leading term decides both ends

Far from 0, the leading term is so much bigger than the other terms that the polynomial goes wherever the leading term goes. An even power is positive on both sides, so both ends point the same way. An odd power has the sign of x, so the ends point opposite ways. A negative leading coefficient flips the picture.



	leading coefficient > 0	leading coefficient < 0
even degree	both ends rise	both ends fall
odd degree	left falls, right rises	left rises, right falls

2. The degree caps the zeros and the turns

A polynomial of degree n has **at most** n **real zeros** and **at most** $n - 1$ **turning points**.

"At most" is a limit, not a promise. $x^2 + 1$ has degree 2 and no real zeros at all.

3. Domain, range and symmetry

Every polynomial has domain all real numbers.

- **Odd degree:** the range is all real numbers.
- **Even degree:** the range stops on one side: $y \geq m$ (a least value) if the leading coefficient is positive, $y \leq M$ (a greatest value) if it is negative.
- **Only even powers** (a constant counts as even): an even function. **Only odd powers:** an odd function. **Both:** neither.

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WATCH OUT

- **Collect like terms before you read the leading term.** Top powers can cancel. In $(x^2+2x)^2 - x^4$ the x^4 terms cancel, so the degree is 3, not 4.
- **In a product, multiply the leading terms of the factors.** The leading term of $(1-3x)(x+4)^2$ is $(-3x)(x^2) = -3x^3$.
- **The degree does not decide symmetry.** $x^3 + 1$ has odd degree, but it is neither even nor odd.
- **The ends say nothing about the middle.** They do not tell you where the zeros are.

WORKED EXAMPLES

Example 1. Describe the end behavior of $f(x) = 6 + x^2 - 3x^4$ and of $g(x) = x^2(5-2x)$.

1. In f the highest power is x^4, so the leading term is $-3x^4$: even degree, negative leading coefficient.
2. In g, multiply the leading terms of the factors: $x^2 \cdot (-2x) = -2x^3$. Odd degree, negative leading coefficient.

Answer: as $x \to \pm\infty$, $f(x) \to -\infty$. As $x \to -\infty$, $g(x) \to \infty$, and as $x \to \infty$, $g(x) \to -\infty$. **Check:** $g(10) = 100 \cdot (-15) = -1500$, very negative far to the right.

Example 2. Show that $c(x) = x^3 - 9x$ has three real zeros and exactly two turning points.

1. Take out the common factor, then use the difference of squares: $x^3 - 9x = x(x^2-9) = x(x-3)(x+3)$.
2. A product is 0 only when a factor is 0, so the zeros are −3, 0 and 3. That is the most degree 3 allows.
3. Test one input in each gap: $c(-1) = -1 + 9 = 8$ and $c(1) = 1 - 9 = -8$.
4. Between −3 and 0 the graph leaves the axis, goes up to 8 and comes back, so it turns at least once. Between 0 and 3 it goes down to −8 and back: another turn.
5. At least two turns, and degree 3 allows at most two.

Answer: zeros −3, 0, 3 and exactly two turning points. **Check:** $c(3) = 27 - 27 = 0$.

Example 3. Find the range of $p(x) = x^4 - 6x^2 + 4$.

1. Even degree, positive leading coefficient: both ends rise, so the range has a least value.
2. Treat x^2 as the variable and complete the square. Half of −6 is −3, and $(-3)^2 = 9$:
$$p(x) = (x^2-3)^2 - 9 + 4 = (x^2-3)^2 - 5$$
3. A square is never negative, so $p(x) \geq -5$. It equals −5 when $x^2 = 3$, at $x = \pm\sqrt{3}$.

Answer: the range is $y \geq -5$. **Check:** expand: $(x^2-3)^2 - 5 = x^4 - 6x^2 + 9 - 5$, which is $x^4 - 6x^2 + 4$.

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