The difference quotient

The average rate of change works on one interval at a time. Can one expression give it for every interval at once?

WORDS TO KNOW

Width h: the step from x to $x + h$. It may be positive or negative, but never 0.

Secant line: the straight line through two points of a graph.

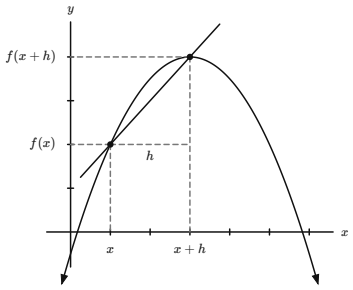
Difference quotient: the average rate of change of f from x to $x + h$, written as a single expression in x and h.

Conjugate: for $A - B$, the expression $A + B$. Their product is $A^2 - B^2$.

KEY IDEAS

1. One expression for every interval

On the interval from x to $x + h$ the outputs are $f(x)$ and $f(x + h)$, and the width is h. Change in output over change in input is the difference quotient. On the graph it is the slope of the secant line through the two points.



$$\frac{f(x+h) - f(x)}{h}, \qquad h \neq 0$$

To use it for one interval, put in its left end for x and its width for h.

2. Three moves: evaluate, subtract, divide

1. **Evaluate at $x + h$:** put the whole of $x + h$ wherever x stands in the rule.
2. **Subtract $f(x)$:** for a polynomial, every term without h cancels.
3. **Divide by h:** every term left has a factor h, so it cancels, since $h \neq 0$.

The simplified form, such as the answer to Example 1 below, has no h alone in the denominator, so it can be worked out for any small h. For a line, the quotient is just the slope, with no x and no h left.

3. Fractions and square roots hide the factor h

For a fraction rule, subtract over a common denominator first. Then the h shows up in the numerator.

For a square root rule, multiply the top and bottom by the conjugate of the numerator. Then $(A - B)(A + B) = A^2 - B^2$ clears the roots from the top.

The result holds only where both ends, x and $x + h$, are in the domain of f.

WATCH OUT

- $(x + h)^2$ **is not** $x^2 + h^2$**.** It is $x^2 + 2xh + h^2$. The middle term is where most of the answer comes from.
- **Subtract all of $f(x)$.** Put $f(x)$ in parentheses, so the minus sign reaches every term.
- **The width cannot be 0.** At $h = 0$ the quotient is $\frac{0}{0}$, undefined: one point makes no interval.
- **If a term without h is left after you subtract a polynomial, you made a slip.** Go back and check the expansion.

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WORKED EXAMPLES

Example 1. Find and simplify the difference quotient of $f(x) = 4x^2 - 7x + 2$.

1. Evaluate at $x + h$, using $(x + h)^2 = x^2 + 2xh + h^2$:
$$f(x + h) = 4x^2 + 8xh + 4h^2 - 7x - 7h + 2$$
2. Subtract $f(x) = 4x^2 - 7x + 2$. The terms $4x^2$, $-7x$ and 2 cancel, leaving $8xh + 4h^2 - 7h$.
3. Every term has a factor h. Divide by h, since $h \neq 0$:
$$\frac{f(x + h) - f(x)}{h} = \frac{h(8x + 4h - 7)}{h} = 8x + 4h - 7$$

Answer: $8x + 4h - 7$. **Check:** on $[1, 3]$, $x = 1$ and $h = 2$ give 8 + 8 − 7 = 9. Directly, $f(1) = -1$ and $f(3) = 17$, and 18 ÷ 2 = 9.

Example 2. Find and simplify the difference quotient of $f(x) = \dfrac{2}{x + 5}$.

1. Evaluate at $x + h$: $f(x + h) = \frac{2}{x+h+5}$.
2. Subtract over the common denominator $(x + h + 5)(x + 5)$. The numerator is
$$2(x + 5) - 2(x + h + 5) = -2h$$
3. Divide by h. The h cancels:
$$\frac{f(x + h) - f(x)}{h} = \frac{-2}{(x + h + 5)(x + 5)}$$

Answer: the expression in step 3, for $x \neq -5$ and $x + h \neq -5$. **Check:** with $x = -3$ and $h = 1$, $f(-3) = 1$ and $f(-2) = \frac{2}{3}$, so the rate is $-\frac{1}{3}$. The formula gives $\frac{-2}{3 \cdot 2} = -\frac{1}{3}$.

Example 3. Find and simplify the difference quotient of $f(x) = \sqrt{2x + 5}$.

1. The quotient is below. The factor h is hidden inside the roots.
$$\frac{\sqrt{2x + 2h + 5} - \sqrt{2x + 5}}{h}$$
2. Multiply the top and bottom by the conjugate $\sqrt{2x + 2h + 5} + \sqrt{2x + 5}$. The top becomes
$$(2x + 2h + 5) - (2x + 5) = 2h$$
3. Cancel the h:
$$\frac{f(x + h) - f(x)}{h} = \frac{2}{\sqrt{2x + 2h + 5} + \sqrt{2x + 5}}$$

Answer: the expression in step 3. It needs $2x + 5 \geq 0$ and $2x + 2h + 5 \geq 0$, not both 0. **Check:** with $x = -2$ and $h = 4$, $f(-2) = 1$ and $f(2) = 3$, so the rate is $\frac{2}{4} = \frac{1}{2}$. The formula gives $\frac{2}{3+1} = \frac{1}{2}$.

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