Estimating a rate at an instant

An average rate needs two inputs. How fast is something changing at one single instant, such as a speedometer reading?

WORDS TO KNOW

Right estimate: the average rate on $[a, a+h]$ with $h > 0$, an interval just to the right of a.

Rate of change at a (instantaneous rate): the one number that the right and the left estimates both approach as h shrinks toward 0.

Tangent line: the line that touches a smooth curve at a point and runs in its direction.

Left estimate: the same quotient with $h < 0$, the interval $[a+h, a]$ just to the left of a.

Centered average: the average rate on $[a-h, a+h]$, an interval sitting evenly around a.

KEY IDEAS

1. Shrink the interval from both sides

For a small width $h \neq 0$, each estimate is a difference quotient at a. Make h smaller and smaller, on the right and on the left. If both lists settle on the same number, that number is the rate at a.

$$\text{estimate} = \frac{f(a+h) - f(a)}{h}$$

If, as h shrinks, the estimates on one side fall and those on the other side rise, both toward the same number, that number is the rate. It lies between every left estimate and every right estimate.

If the two sides approach different numbers, there is no rate. The graph has a sharp corner there, like $|x|$ at 0.

2. The rate is the slope of the tangent line

Each estimate is the slope of a secant through $P = (a, f(a))$. As h shrinks, the secant pivots toward the tangent line at P.

If f has rate r at a, the tangent line is $y = f(a) + r(x - a)$. Near a it gives good estimates of f.

3. The centered average is usually more accurate

$$\text{centered average} = \frac{f(a+h) - f(a-h)}{2h}, \qquad h > 0$$

It is the mean of the left and the right estimate of the same width. For a quadratic it equals the rate exactly, for every h. For a line, every estimate equals the slope.

The left and right errors are nearly equal and opposite, so their mean cancels most of them.

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WATCH OUT

- **Do not put $h = 0$ into the quotient.** That gives $\frac{0}{0}$, not the rate. Every estimate needs a width that is not 0.
- **Check both sides before you trust a centered average.** At a corner the centered averages can settle on the mean of the two sides, a slope that neither side has.
- **A table limits the width.** You cannot use a width smaller than the spacing of the table, and rounded readings make very tiny widths useless.
- **The function must exist on both sides of a.** At the end of a domain, like $x = 0$ for $\sqrt{x}$, there is no rate in this sense.

WORKED EXAMPLES

Example 1. Estimate the rate of change of $f(x) = x^3 + 18x$ at $x = -2$. Then use the tangent line there to estimate $f(-1.9)$.

1. $f(-2) = -44$. With $(-2 + h)^3 = -8 + 12h - 6h^2 + h^3$, $f(-2 + h) - f(-2) = 30h - 6h^2 + h^3$.
2. Divide by $h \neq 0$: each estimate is $30 - 6h + h^2$.
3. Right: $h = 0.1$ gives 29.41, and $h = 0.01$ gives 29.9401. Left: $h = -0.1$ gives 30.61, and $h = -0.01$ gives 30.0601. Both sides head to 30, so the rate is 30.
4. Tangent line: $y = -44 + 30(x + 2)$, which is $y = 30x + 16$. At $x = -1.9$ it gives −41.

Answer: the rate is 30, and $f(-1.9) \approx -41$. **Check:** the true value is $(-1.9)^3 + 18(-1.9)$, which is $-6.859 - 34.2 = -41.059$, close to the estimate.

Example 2. A cyclist's distance from the start is logged each second. Estimate her speed at $t = 2$.

t (seconds)	0	1	2	3	4
distance (meters)	0.0	3.2	7.0	11.4	16.4

1. The narrowest width the table allows is $h = 1$. Right estimate: $11.4 - 7.0 = 4.4$. Left estimate: $7.0 - 3.2 = 3.8$. They disagree: she is speeding up.
2. Centered, $h = 1$: (11.4 − 3.2) ÷ 2 = 4.1, the mean of 4.4 and 3.8.
3. Centered, $h = 2$: (16.4 − 0.0) ÷ 4 = 4.1. The two centered averages agree.

Answer: about 4.1 meters per second at $t = 2$.

Example 3. Does $g(x) = |x^2 - 16|$ have a rate of change at $x = 4$?

1. $g(4) = 0$. For $h > 0$, $(4 + h)^2 > 16$, so $g(4 + h) = 8h + h^2$ and each right estimate is $8 + h$.
2. For $h < 0$ near 0, $(4 + h)^2 < 16$, so the absolute value flips the sign: $g(4 + h) = -8h - h^2$, and each left estimate is $-8 - h$.
3. The right estimates approach 8 and the left estimates approach −8.

Answer: no. The two sides approach different numbers, so the graph has a corner at $x = 4$. (The centered averages here are all h, which head to 0 and hide the corner.)

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